Mathematics for Computer Science
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Induction III
Strong Induction
Least Number Principle

Strong Induction

Allows proving $P(n)$ from *all* of
$P(0), P(1), \ldots, P(n-1)$,
instead of just $P(n-1)$.

Rosen calls Strong Induction:
The *Second Principle* of Induction

Strong Induction

Nothing new: Ordinary Induction,
started at 0, then
$0 \to 1,\ 1 \to 2,\ 2 \to 3,\ \ldots,\ n \to n+1$.
So by the time you get to $n+1$,
you already got to all $k < n+1$.

Strong Induction

Note:
By convention, strong induction target is n rather than $n+1$:

$0 \to 1,\ 1 \to 2,\ 2 \to 3,\ \ldots,\ n-1 \to n$.

Strong Induction

If n is red when everything $< n$ is red
Then everything is red.

$$\frac{[\forall k < n\ \ R(k)] \to R(n)}{\forall n\ \ R(n)}$$

Strong Induction

In strong induction, the base case may be built into the induction step:

At $n = 0$, Strong Induction step allows
assuming $P(k)$ for all $k < n = 0$.
There aren't any such $k \in$ ¢ !

Strong Induction

So you have to prove $P(0)$ assuming nothing – acts like a base case but appears within the induction step.

Strong Induction

Example:  stamps

5¢ 3¢

Theorem: can form any amount ≥ 8¢

Prove by strong induction on n.

$P(n)$ ::= if $n \geq 8$, can form n¢.

(Picture source:http://site17585.dellhost.com/lsj/facts/s_events.htm http://www.frbsf.org/currency/civilwar/stamps/s150.html)

Strong Induction

Assume m¢ for all $m < n$ in order to prove n¢.

Proof by cases:

$n < 8$¢: *vacuously* true

$n = 8$¢: 

Strong Induction

Proof by cases:

$n = 9$¢: 

$n = 10$¢: 

Strong Induction

Case $n \geq 11$¢:

Now $n > n - 3 \geq 8$, so

by Strong Induction we have:

 +  = n ¢!

$n - 3$ add stamp

QED

Strong vs. Ordinary Induction

MetaTheorem: Can transform any Strong Induction proof into Ordinary Induction.

We'll illustrate with the 8¢ example, but simple transformation always works for *any* example.

Strong vs. Ordinary Induction

Theorem: Can form any amount ≥ 8¢

Reprove by *ordinary* induction using induction hypothesis:

$Q(n) ::= \forall k \leq n \; P(k)$

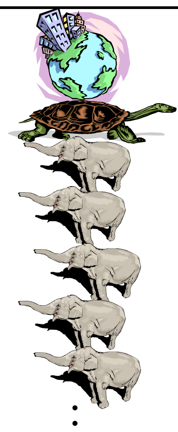
$::= \forall k \leq n$ if $k \geq 8$, can form k¢.

Earlier Strong Induction now goes through by Ordinary Induction.

Strong vs. Ordinary Induction

So why use Strong?

-- Convenience: no need to include $\forall k \leq n$ all over.



Problems

Class Problems 1 & 2

Least Number Principle

Called *Well-ordering Property* by Rosen

Least Number Principle

Every nonempty set of
nonnegative integers
has a
least element.

Familiar? Now you mention it, Yes.
Obvious? Yes.
Trivial? Yes. But watch out:

Least Number Principle

Every nonempty set of
nonnegative rationals
has a
least element.

NO!

Least Number Principle

Every nonempty set of
~~*nonnegative*~~ *integers*
has a
least element.

NO!

Least Number Principle

Theorem: Every integer > 1 is a product of primes.

Prove using LNP.

Proof: (by contradiction) Suppose not.
Then set of nonproducts is nonempty.
By LNP, there is a least $n > 1$ that is not a product of primes.
In particular, n is not prime.

Least Number Principle

Theorem: Every integer > 1 is a product of primes.

Proof: …So $n = k \cdot m$ for integers k, m where $n > k,m > 1$.
Since k,m smaller than the *least* nonproduct, both are prime products, *eg.*,

$$k = p_1 \cdot p_2 \cdots p_{94}$$
$$m = q_1 \cdot q_2 \cdots q_{214}$$

Least Number Principle

Theorem: Every integer > 1 is a product of primes.

…So

$$n = k \cdot m = p_1 \cdot p_2 \cdots p_{94} \cdot q_1 \cdot q_2 \cdots q_{214}$$

is a prime product, a contradiction.

∴ The set of nonproducts > 1 must be empty. QED

Class Problems 3 & 4