Mathematics for Computer Science

MIT 6.042J/18.062J

Predicates & Quantifiers

Induction

Predicates

Predicates are

Propositions with variables

Example:

$$P(x,y) \underbrace{::=}_{\text{“is defined to be”}} x + 2 = y$$

Predicates

$$P(x,y) ::= x + 2 = y$$

$x = 1$ and $y = 3$: $P(1,3)$ is true

$x = 1$ and $y = 4$: $P(1,4)$ is false

$\neg P(1,4)$ is true

Quantifiers

$\forall x$ For ALL x

$\exists y$ There EXISTS some y

Quantifiers

x, y range over ***Domain of Discourse***

$$\forall x\, \exists y\;\; x < y$$

Domain	*Truth value*
integers $\mathbb{Z}$	True
positive integers $\mathbb{Z}^+$	True
negative integers $\mathbb{Z}^-$	False
negative reals $\mathbb{R}^-$	True

Validity

$$[\forall x \forall y\ Q(x,y)] \rightarrow \forall z\ Q(z,z)$$

True ***no matter what***

- the Domain is,
- predicate Q is.

Problems

Class Problems 1& 2

Proof by Induction

An Example of Induction

Suppose we have a property (say *color*) of the natural numbers:

0, 1, 2, 3, 4, 5, …

Showing that *zero is red*, and that

the *successor of any red number is red*,

proves that *all numbers are red*!

The Induction Rule

0 and (from n to n+1)

proves 0, 1, 2, 3,….

$$\frac{R(0), \forall n \in \mathbb{N}\ [R(n) \rightarrow R(n+1)]}{\forall m \in \mathbb{N}\ R(m)}$$

Proof by Induction

Statements in green form a template for inductive proofs:

Proof: (by induction on n)

The induction hypothesis:

$$P(n) ::= \quad 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

An Aside: Ellipses

Ellipses (…) mean that the reader is supposed to *infer* a pattern.

- This can lead to confusion about what is being stated.
- Here summation notation gives more precision, for example:

$$1 + r + r^2 + \cdots + r^n = \sum_{i=0}^{n} r^i$$

Example Induction Proof

Base Case ($n = 0$):

$$\underbrace{1 + r + r^2 + \cdots + r^0}_{1} \overset{?}{=} \frac{r^{0+1} - 1}{r - 1} = \frac{r-1}{r-1} = 1$$

Wait: divide by zero bug!

This is only true for $r \neq 1$

An Example Proof

Revised Induction Hypothesis:

$$P(n) ::= \forall r \neq 1 \;\; 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

An Example Proof

Induction Step: Assume $P(n)$ for $n \geq 0$ to prove $P(n + 1)$:

$$\forall r \neq 1 \;\; 1 + r + r^2 + \cdots + r^{n+1} = \frac{r^{(n+1)+1} - 1}{r - 1}$$

An Example Proof

Have $P(n)$ by assumption:

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Adding r^{n+1} to both sides:

$$1 + \cdots + r^n + r^{n+1} = \frac{r^{n+1} - 1}{r - 1} + r^{n+1}$$

$$= \frac{r^{n+1} - 1 + r^{n+1}(r-1)}{r - 1}$$

An Example Proof

Continued…

$$1 + \cdots + r^n + r^{n+1} = \frac{r^{n+1} - 1 + r \cdot r^{n+1} + -r^{n+1}}{r - 1}$$

$$= \frac{r^{(n+1)+1} - 1}{r - 1}$$

Which is just $P(n+1)$
QED.

Problems

Class Problem 3