*Mathematics for Computer Science*

**MIT 6.042J/18.062J**

# Milestones of Probability

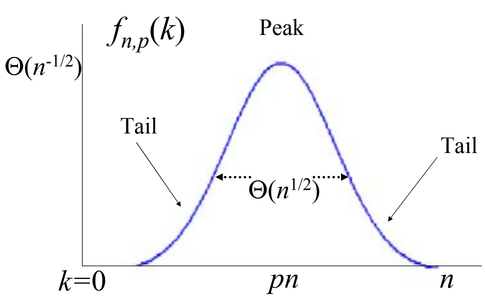



## 3 Milestones

- Central Limit Theorem
- Poisson Distribution
- Strong Law of Large Numbers




## The Binomial PDF







## Binomial Limits

Is there a "limiting shape" for $f_{n,p}$?

What limit? – two parameters:

- $p$ fixed, $n \to \infty$
- ~~$n$ fixed, $p \to 0$ or 1~~
- $n \to \infty$, $p \to 0$ together  how?




## Binomial Limits

Is there a "limiting shape" for $f_{n,p}$?

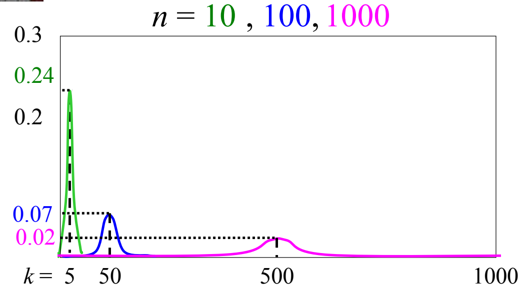
- $p$ fixed, $n \to \infty$

  Say $p = 1/2$



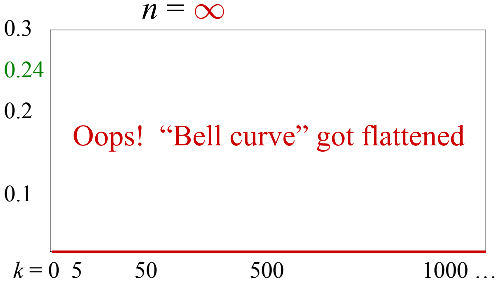

## Fair ($p = 1/2$) Binomial Distribution

$n = 10$, 100, 1000



## Fair ($p = 1/2$) Binomial Distribution

$n = \infty$

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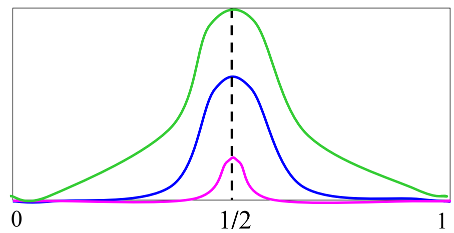
## Shift to same Mean and Scale [0,1]

($x$-axis × $1/n$)

$n$ = 10 , 100, 1000

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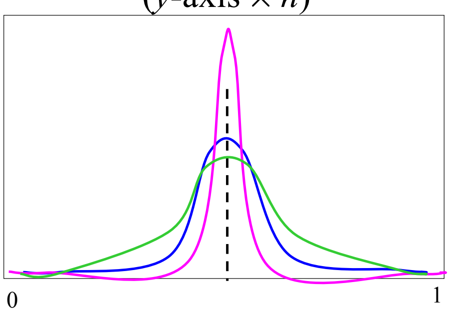
## Scale to Same *Area*

($y$-axis × $n$)

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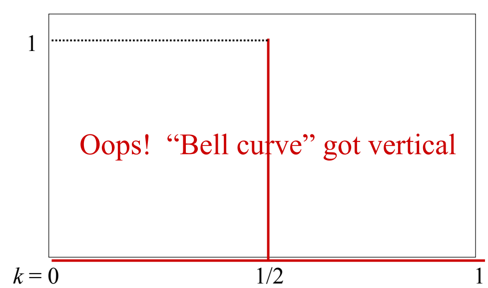
## Fair ($p = 1/2$) Binomial Distribution

$n = \infty$

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## Normalize!

- Shift to expectation 0 (subtract $\mu_n$)
- Scale $x$-axis to SD 1 (divide by $\sigma_n$)

$$B^* ::= \quad (B-\mu)/\sigma$$

- Scale $y$-axis to area 1 (multiply by $\sigma_n$) to see Bell Curve.

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## Central Limit Theorem

As $n \to \infty$, $p$ fixed,

$$B^*_{n,p} \quad \to \text{Bell-shaped Curve}$$

technically:

$$\eta(x) ::= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\lim_{n\to\infty} \sigma_n f_{n,p}(\alpha n) = \eta\left(\frac{\alpha n - \mu_n}{\sigma_n}\right)$$

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## Central Limit Theorem

Could derive η from Stirling's formula (see Feller). Other proofs along lines of Chernoff bound proof (using "moment generating function").

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## Central Limit Theorem

More generally, for any independent, identically distributed

$$X_1, X_2, \cdots, X_n, \cdots$$

let $S_n ::= X_1 + X_2 + \cdots + X_n$

As $n \to \infty$, $p$ fixed,

$$S_n^* \to \text{Bell Curve}$$

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## In-Class Problem

# Problem 1

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## Poisson Distribution

Misprints in Weekly Notes: $\lambda \approx 1/3$ per page.
30 lines × 90 char,

$n \approx 2700$ char/page.

Say each char has *independent* prob

$p ::= \lambda/n = 1/3 \cdot 1/2700 = 1/8100$

of misprint. pdf for errors/page would be:

$$f_{n,\lambda/n} = f_{2700,1/8100}$$

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## Poisson Distribution

Packets at router: $\lambda \approx 10$/μsec.

$n = 1000$ cycles/μsec, $\lambda/n = \Pr\{\text{packet at cycle}\}$

Say each cycle has *independent* prob of containing a packet: pdf for packets/μsec would be:

$$f_{n,\lambda/n} = f_{1000,1/100}$$

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## Poisson Distribution

Bombs/rescue team:

$\lambda \approx$ 200 bombers × 50 bombs/1500 rescue teams

$n = 1500$

Say bomb has *independent* prob of landing in team area: pdf for bombs/team would be:

$$f_{n,\lambda/n} = f_{1500,2/3}$$

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## Poisson Distribution

Common theme:
Moderate average, $\lambda$, arising from a large number, $n$, of unlikely samples

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## Poisson Limit Theorem

As $n \to \infty$, $p \to 0$, $np = \lambda$

$B^*_{n,\lambda/n} \to$ Poisson var $P_\lambda$

- one parameter $\lambda$
- $\mu = \lambda$
- $\sigma = \sqrt{\lambda}$

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## Poisson Distribution

$$\Pr\{P_\lambda = k\} ::= \frac{\lambda^k}{k!} e^{-\lambda}$$

Routine proof using
$\lim_{n \to \infty} (1 + x/n)^n = e^x$

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## Strong Law of Large Numbers

- Random Var $R$, unknown $\mu_R$
- Keep sampling $R_1, R_2, \ldots$
- intuitively expect

$$\frac{R_n}{n} \overset{?}{\longrightarrow} \mu_R$$

True – with probability 1,
but does NOT follow from Weak Law

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## In-Class Problem

# Problem 2

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