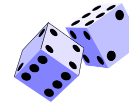
Mathematics for Computer Science

MIT 6.042J/18.062J

Great Expectations.III

Sum of 3 Dice



- Roll 3 fair dice
- What is the expected sum?

$R = R_1 + R_2 + R_3$

$E[R] = E[R_1] + E[R_2] + E[R_3]$

$= 3.5 \cdot 3$

Sum of 2 or 3 Dice

- Flip a fair coin
  - If heads, roll 3 fair dice
  - If tails, roll 2 dice
- What is the expected sum?

$R = R_1 + R_2 + R_3$ or $R_1 + R_2$

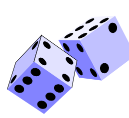
Conditional Expectation

$$E[R \mid A] = \sum_r r . \Pr\{R = r \mid A\}$$

Law of Total Expectation

$$E[R] = \sum_i E[R \mid A_i] \cdot \Pr\{A_i\}$$

Sum of 2 or 3 Dice



If heads, roll 3 fair dice,

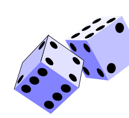
if tails, roll 2 dice

$E[R] = E[R \mid \text{heads}].\Pr\{\text{heads}\} + E[R \mid \text{tails}].\Pr\{\text{tails}\}$

$E[R] = (3.5 \times 3).\tfrac{1}{2} + (3.5 \times 2).\tfrac{1}{2}$

$= 3.5 \times 2.5$

Roll a Dice,



- If = 1, then roll 1 dice
- If = 2, then roll and sum 2 dice
- If = 3, then roll and sum 3 dice
- …..
- If = 6, then roll and sum 6 dice

- What's the expected sum?

Wald's Theorem

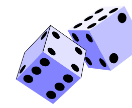
Let $R = R_1 + R_2 + R_3 + R_4 + \cdots + R_Q$

where Q is also a random variable

Then $E[R] = E[R_i] \times E[Q]$

(many restrictions apply)

Sum of 2 or 3 Dice



If heads, roll 3 fair dice,

if tails, roll 2 dice

$E[R_i]$ = expected value of a dice = 3.5

$E[Q]$ = expected number of rolls = 2.5

By Wald's Theorem

$E[R] = 3.5 \times 2.5$

Same answer!

Useful Way for Analyzing Processes

Leader Election

Expected Time

= Time taken/round . E[#Rounds]

(FIXED) $1/p_{success}$

Wald's Theorem

Different process:

Time taken/round is not fixed

Expected Time

=E[Time taken/round] . E[#Rounds]

Example:

- Flip a coin until get 3 heads in a row
- Possible run: HTTHHTHTHHH
- How many coin flips on average?

Example:

- Possible run: HTTHHTHTHHH

How many rounds did I run? $E[Q]$=? $=1/p_{success}$

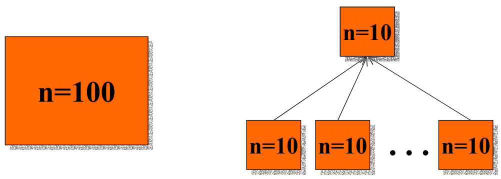
1: HT
2: T
3: HHT
4: HT
5: HHH

How many coin flips before I get a T?

$E[R_i]$=?

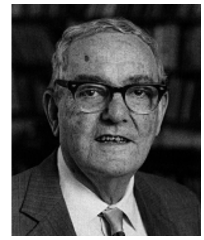
A Case for Modular Design

Let T_n be the time to assemble n parts



$E[T_{100}]$ $E[T_{100}] = 10.\ E[T_{10}]$

Herbert Simon



The Architecture of Complexity, 1962

In Class Problem 1:

Find the Bug!!!

Fun with Infinite Series

Consider the Series:

- 1 + -1 + 1 + -1 + 1 + -1…….
- (1 + -1) + (1 + -1) + (1 + -1)……. = 0

But I could rearrange the grouping,

- 1 + (-1 + 1) + (-1 + 1) + (-1 + 1)… = 1 !!!

= 2

= 100

Infinite Linearity of Expectations

$E[C_1 + C_2 + C_3 + \ldots.]$

$= E[C_1] + E[C2] + E[C_3] \ldots\ldots$

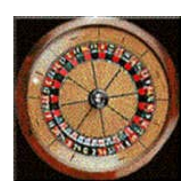
But!!! It might converge to different values based on rearranging the terms!

(must check for absolute convergence) $\sum_{i=1}^{\infty} E[|Ci|]$

The Bet-Doubling Strategy

At any round r

- Place a bet = $\$10 \cdot 2^{r-1}$
- Total bet so far:
- = $\$10\ (2^0 + 2^1 + 2^2 + \ldots + 2^{r-1})$
- = $\$10\ (2^r - 1)$



- If you win, receive: $\$10 \cdot 2^r$
- Net Win = $10

What is the Expected Win

- Correct Argument
  - Expected number of rounds = $1/p = 2$
  - But probability of never getting red = 0
  - Win $10 with certainty
- Incorrect Argument
  - Expected win in one round $E[C_i] = 0$
  - Therefore expected win $E\left[\sum_{r=1}^{\infty} C_i\right] = 0$

Problem: What's your worth?

- Let M = amount of money you had to put up
- At round r, $M = \$10\,(2^r - 1)$
- Pr{win round on r} $= (1-p)^{r-1}p = 1/2^r$

$$E[M] = \sum_{r=1}^{\infty} 10 \cdot (2^r - 1) \cdot \frac{1}{2^r} \approx \sum_{r=1}^{\infty} 10$$

= infinite

In Class Problems

In Class Problem 1 (c)

E[win] = ½ + ½ + ½ + ½ +… = ∞.

What if the Casino is limited to a Billion dollars (2^{30})?

$$E[win] = \sum_{i=1}^{30} 2^i \frac{1}{2^{i-1}} + \frac{1}{2^{30}} \cdot 2^{30}$$

(< 30 *tails*) (30 *tails*)

= 15 + 1 = 16!!!!

Infinite Expectation

- Random Variable, R
- Probability Density, $f_R(r) ::= \Pr\{R=r\}$

$$\sum_{r \in range(R)} \Pr\{R = r\} = 1$$

← MUST BE TRUE

$$E[R] ::= \sum_{r \in range(R)} r \cdot \Pr\{R = r\}$$

← MIGHT BE INFINITE

Example:

$R = 2, 4, 8, 16, 32, \ldots$

where

$$\Pr\{R = 2^i\} = \frac{1}{2^i}$$

$$\sum_{i=1}^{\infty} \Pr\{R = 2^i\}$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i}$$

$$= \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i - 1$$

$$= 2 - 1 = 1$$

$$\mathrm{E}[R]$$

$$= \sum_{i=1}^{\infty} r \Pr\{R = r\}$$

$$= \sum_{i=1}^{\infty} 2^i \frac{1}{2^i}$$

$$= \sum_{i=1}^{\infty} 1 \quad \longleftarrow \textit{infinite}$$