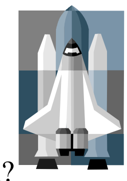
Mathematics for Computer Science

MIT 6.042J/18.062J

Great Expectations.II

Space Station Mir

Suppose that main computer has a probability p of failing every second



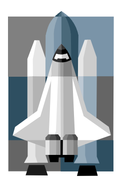
- When do we expect it to first fail?

Space Station Mir

T = time of the first failure

Compute $E[T]$

$\Pr\{T=i\}$ = probability it first fails on the i^{th} step



Probability Density Function

$\Pr\{\text{fails in the } 1^{st} \text{ second } (T=1)\} = p$

$\Pr\{\text{fails in the } 2^{nd} \text{ second } (T=2)\} = (1-p)\cdot p$

$\Pr\{\text{fails in the } 3^{rd} \text{ second } (T=3)\}$
$= (1-p)\cdot(1-p)\cdot p$

Probability Density Function

Probability that it fails in i^{th} second

$$\Pr\{T=i\} = (1-p)^{i-1}\cdot p$$

↑ no failures in first i-1 seconds

Verify!

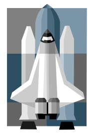
The probability density function should sum to 1!

$$\sum_{i=1}^{\infty} \Pr\{T=i\}$$

$$= \sum_{i=1}^{\infty} (1-p)^{i-1} p$$ ← Geometric Series

$$= p.\frac{1}{1-(1-p)} = \frac{p}{p} = 1$$ ✓

Space Station Mir



$$E[T] = \sum_{i=0}^{\infty} i \cdot \Pr\{T = i\}$$

$$= \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} p$$

← Again a series we can deal with!

$$= \frac{1}{p}$$

← Mean Time to Failure

Reminder

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

If we differentiate both sides

$$\sum_{i=0}^{\infty} i \cdot x^{i-1} = \frac{1}{(1-x)^2}$$

More About Mir

- How many times should I expect to roll a fair dice before I get a 6?
- Mir contains 10,000 components each with a probability p of failing.
  - How many failures do we expect to see?

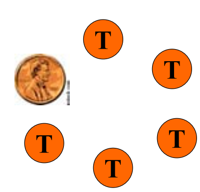
In Class Problem 1

Write solutions on the board

Scapegoat Election

Scapegoat Election

- Everyone flips a coin
- Person who flips H is the scapegoat



- If none or multiple Hs, repeat the process.

In Class Experiment

Question: How long does it take to elect a scapegoat?

Analysis

How long does it take to elect a scapegoat?

1. What is the probability of electing a scapegoat in a given round?
2. What is the expected number of rounds before a scapegoat is successfully elected?

Success in a Given Round

Given n people,
Pr{goat is chosen in a given round}
= Pr{*exactly* 1 head}

$$= n\left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}\right)$$

HTT
THT
TTH

n positions for the head; n-1 tails; 1 head

Success in a Given round

Given n people,
Pr{goat is chosen in a given round}
= Pr{*exactly* 1 head}

$$= \frac{n}{2^n}$$

How long does it take to elect a scapegoat?

What is the probability of electing a scapegoat in a given round?
– *Answer:* $p_R = \dfrac{n}{2^n}$

What is the expected number of rounds before a scapegoat is successfully elected?

Mean Time to Failure/Success

If given round succeeds with p_R,
then the expected number of rounds
till the first success:

$$E[\#Rounds] = 1/p_R$$

$$= 2^n/n$$

Improvement: Use a Biased Coin

Let Pr{heads} = b,

Pr{tails} = $1-b$.

Pr{*exactly* one head} $= n\,(1-b)^{n-1}\,b$

What happens when b is *large*?

when b is *small*?

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Calculating the Optimal Bias

- Maximizing Pr{success in a given round}:

$$\frac{d}{db}\left(n\,(1-b)^{n-1}\,b\right) = 0$$

Optimal bias $b = 1/n$

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Probability of Success in a Round

Pr{success in a given round}

$$= \left(1 - \frac{1}{n}\right)^{n-1}$$

$p_R \approx 1/e$

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Mean Time to Failure/Success

If given round succeeds with $p_R = 1/e$, then the expected number of rounds till the first success:

E[#Rounds] $= 1/p_R$

$\approx e$ (≈ 2.7 rounds)

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Exercise

How can you simulate a *biased* coin using a *fair* coin?

Say want Pr{heads} $= b = 1/8$?

Or $=1/7$?

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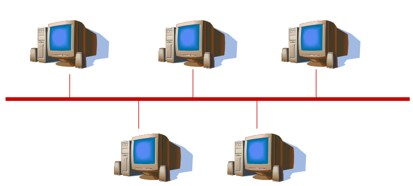
Leader Election and Ethernet

- n computers have to talk on the same coaxial cable, who gets to go first?



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Exponential Backoff

Optimal bias $p = 1/n$, but what if you don't know what n is?

Be optimistic, use $p=1$

- If that fails, $p = 1/2$
- If that fails, $p = 1/4$
- If that fails, $p = 1/8$
- …

Metcalfe and Boggs,1976

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