6.01: Introduction to EECS I

Lecture 9
Modeling Dynamics

April 8, 2008

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Freshman Open House

Friday, April 11, 2008 – 3:30 to 5:00 PM – Room 34-401

For Freshmen: Free T-shirts (while supplies last)
Department Memorabilia
Handouts
LOTS of Food

Here's what will be going on:
Welcome to EECS, Prof. Eric Grimson, Department Head
Short Research Presentations by Faculty:
Prof. Robert C. Miller, "Lightweight Publishing, Automation, and Customization for the Web"
Prof. Vladimir Bulovic, "Lighting Up the World with Quantum Dots"
Prof. Polina Golland, "Understanding Activity Patterns in the Brain"

ALL Freshmen invited, especially potential majors in VI!

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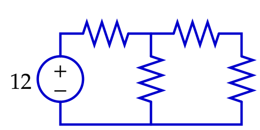
Analyzing Circuits as Constraints: One Ports

Systems of one-port elements can be analyzed by combining three types of constraints:

- KCL
- KVL
- component relations for each one-port
  - resistor: $V = IR$
  - voltage source: $V = V_s$
  - current source: $I = I_s$



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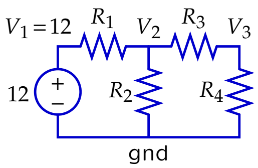
Analyzing Circuits as Constraints: One Ports

The "node" method is one (of many) ways to systematically choose equations and unknowns.

- label all nodes except one (gnd)
- write KCL for each node whose voltage is not known
- solve



KCL at node 2: $\frac{V_2 - 12}{R_1} + \frac{V_2}{R_2} + \frac{V_2 - V_3}{R_3} = 0$

KCL at node 3: $\frac{V_3 - V_2}{R_3} + \frac{V_3}{R_4} = 0$

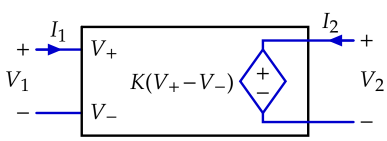
Two equations; two unknowns: V_2 and V_3.

Analyzing Circuits as Constraints: Two Ports

Two-port elements can be represented by two constraints.

Example: op amp

- port 1: $I_1 = 0$
- port 2: $V_2 = K(V_+ - V_-)$

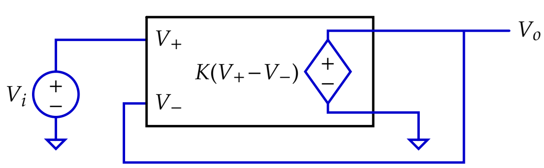


Notice that $I_1 = 0$ constraint can be explicit (as it is above) or implicit, as is done in our software circuit solver (where there is no I_1 variable).

Analyzing Op Amp Circuits

Solving op amp constraints.

Example: non-inverting buffer

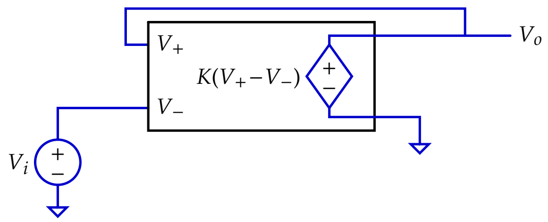


$$V_o = K(V_+ - V_-) = K(V_i - V_o)$$

$$(K + 1)V_o = KV_i$$

$$\frac{V_o}{V_i} = \frac{K}{K + 1}$$

Check Yourself

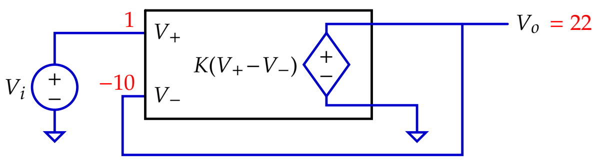


Which of the following equations describes $\frac{V_o}{V_i}$?

1. $\frac{K}{K+1}$
2. $\frac{K}{K-1}$
3. $\frac{-K}{K+1}$
4. $\frac{-K}{K-1}$
5. none of the above

"Thinking" like an op amp

What would the op amp do if the input voltage changed suddenly from 0 V to 1 V? Assume $K = 2$.



This cannot be right!
We used this circuit in lab.
What is wrong with this reasoning?

"Thinking" like an op amp

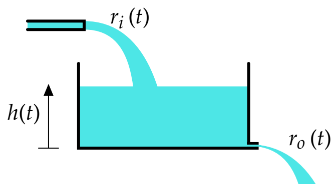
This reasoning is wrong because it ignores a critical property of circuits.

For a voltage to change, charged particles must flow.

To understand flow, we need to understand continuity.

Flows and Continuity

If a quantity is conserved, then the difference between what comes in and what goes out must accumulate.

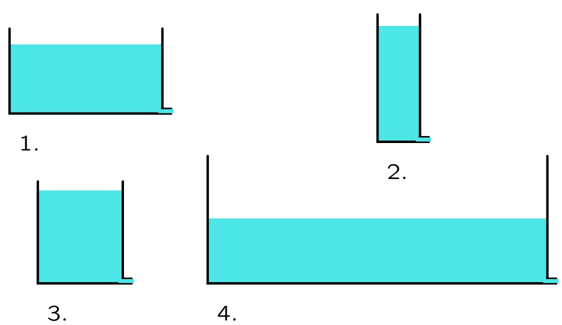


Assume water is conserved: $\frac{dh(t)}{dt} \propto r_i(t) - r_o(t)$

What determines the leak rate r_o?

Check Yourself

The holes in each of the following tanks have equal size. Which tank has the largest leak rate r_o?



Flows and Continuity

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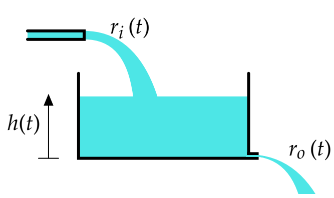


Assume water is conserved: $\frac{dh(t)}{dt} \propto r_i(t) - r_o(t)$

Assume linear leaking: $r_o(t) \propto h(t)$

Solve: $\frac{dr_o(t)}{dt} \propto r_i(t) - r_o(t)$

Check Yourself

$$\frac{dr_o(t)}{dt} \propto r_i(t) - r_o(t)$$

What are the dimensions of the missing constant of proportionality?

Analysis of the Leaky Tank

Call the constant of proportionality $1/\tau$. Then τ is called the **time constant** of the system.

$$\frac{dr_o(t)}{dt} = \frac{r_i(t)}{\tau} - \frac{r_o(t)}{\tau}$$

Analysis of the Leaky Tank

Call the constant of proportionality $1/\tau$. Then τ is called the **time constant** of the system.

$$\frac{dr_o(t)}{dt} = \frac{r_i(t)}{\tau} - \frac{r_o(t)}{\tau}$$

Solve:

Make a discrete time approximation. Assume $r[n] = r(nT)$:

$$\frac{r_o[n+1] - r_o[n]}{T} = \frac{r_i[n]}{\tau} - \frac{r_o[n]}{\tau}$$

Then

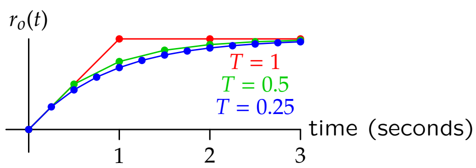
$$r_o[n+1] = \left(1 - \frac{T}{\tau}\right) r_o[n] + \frac{T}{\tau} r_i[n].$$

Analysis of the Leaky Tank

Determine $r_o(t)$ for $t > 0$ assuming that the tank is initially empty and that $r_1(t)$ goes from 0 to 1 at $t = 0$. Let $\tau = 1$ second.

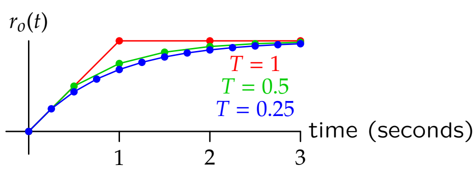
$$r_o[n+1] = \left(1 - \frac{T}{\tau}\right) r_o[n] + \frac{T}{\tau} r_i[n].$$

Try different stepsizes T. Solutions for different values of T converge when T is small compared to the time constant τ.



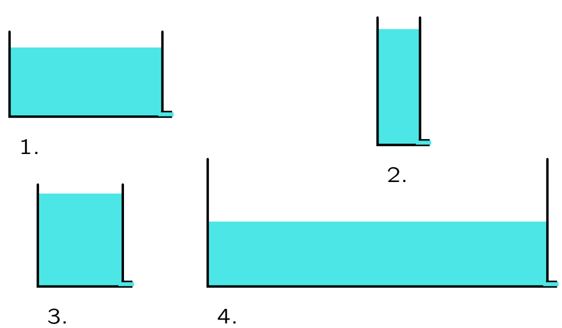
Check Yourself

Provide a physical explanation of the shape of this response.



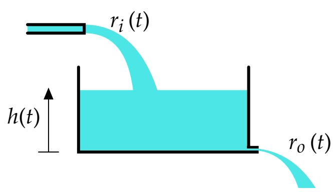
Check Yourself

Which of the following tanks has the largest time constant τ?

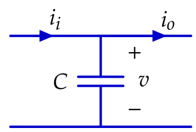


Leaky Tanks and Capacitors

Water accumulates in a leaky tank.



Charge accumulates in a capacitor.



$$\frac{dv}{dt} = \frac{i_i - i_o}{C} \propto i_i - i_o \qquad \text{analogous to} \qquad \frac{dh}{dt} \propto r_i - r_o$$

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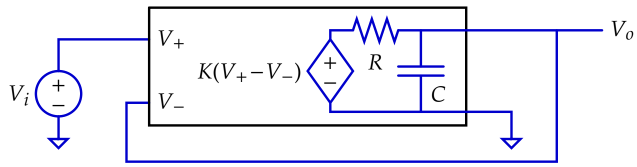
Charge Accumulation in an Op Amp

We can add a resistor and capacitor to "model" the accumulation of charge in an op amp.



R and C are NOT inside an op amp.

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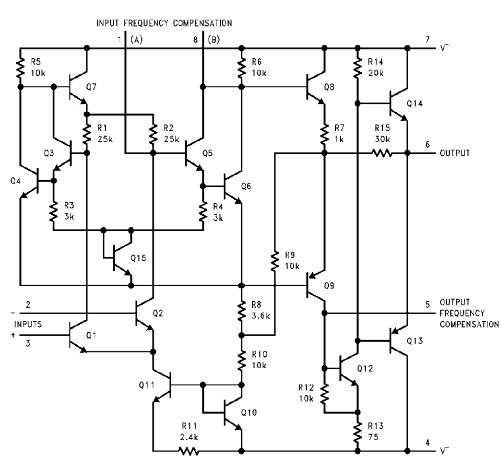
Op Amp Model

Here is a more accurate circuit model of a μA709 op amp.



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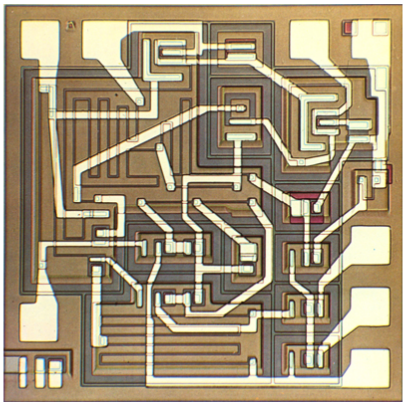
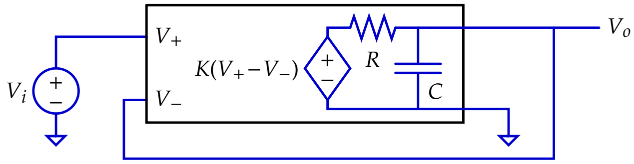
Op Amp

This artwork shows the physical structure of a μA709 op amp.



Charge Accumulation in an Op Amp

We can add a resistor and capacitor to "model" the accumulation of charge in an op amp.



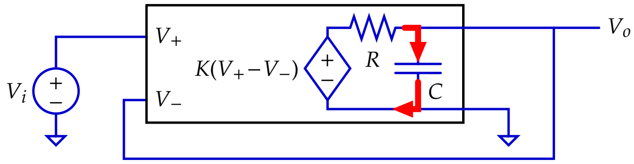
R and C are NOT inside an op amp.

They are parts of our circuit **model** of an op amp.

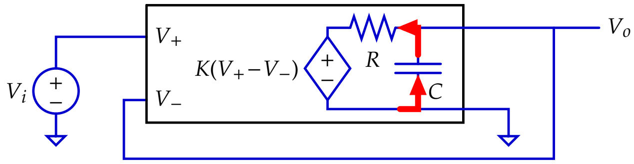
This is an example of using the electical circuit language itself as a modeling language.

Dynamic Analysis of Op Amp

If $V_i > V_o$ then the dependent voltage source adds charge to the capacitor and V_o rises.

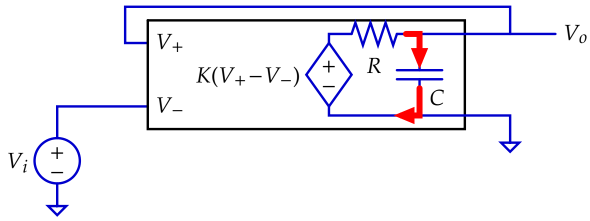


If $V_i < V_o$ then the dependent voltage source removes charge to the capacitor and V_o falls.



Dynamic Analysis of Op Amp

Switching the plus and minus input leads flips these relations. Now if $V_o > V_i$ the dependent voltage source adds charge to the capacitor and V_o rises.

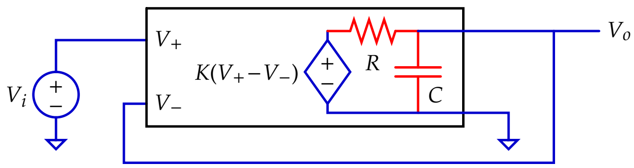


Such systems are said to have "positive feedback."

Positive feedback tends to make systems unstable.

Dynamic Analysis of Op Amp

We can analyze the stability of a circuit by making a block diagram model.



First model the RC circuit. Let $V_2 = K(V_+ - V_-)$. Then the capacitor current I_C is given by

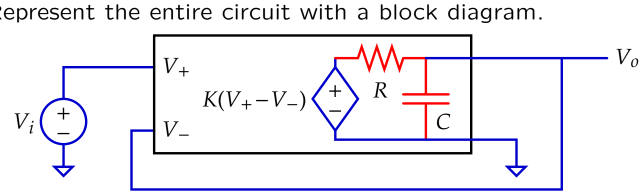
$$I_C = \frac{V_2[n] - V_0[n]}{R} = C\frac{dV_o}{dt} \approx C\frac{V_o[n+1] - V_o[n]}{T}$$

which can be solved for $V_o[n+1]$:

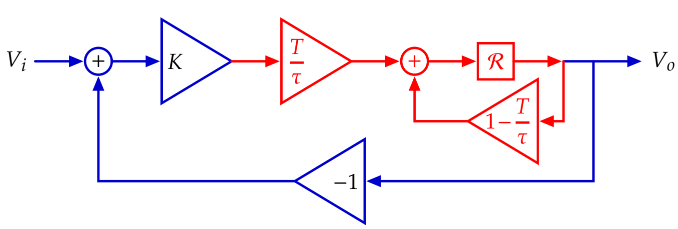
$$V_o[n+1] = \left(1 - \frac{T}{\tau}\right)V_o[n] + \frac{T}{\tau}V_2[n] \qquad \text{where} \qquad \tau = RC$$

Dynamic Analysis of Op Amp

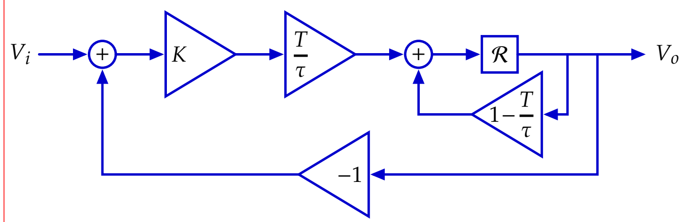
Represent the entire circuit with a block diagram.



$$V_o[n+1] = \left(1 - \frac{T}{\tau}\right)V_o[n] + \frac{T}{\tau}V_2[n]$$



Check Yourself



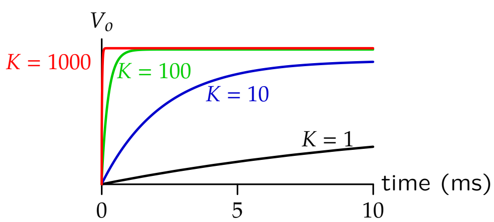
This system has a pole at

1. $1+\frac{T}{\tau}$
2. $1-\frac{T(K-1)}{\tau}$
3. $1-\frac{T(K+1)}{\tau}$
4. $1-\frac{T}{\tau}$
5. none of above

Dynamic Analysis of Op Amp

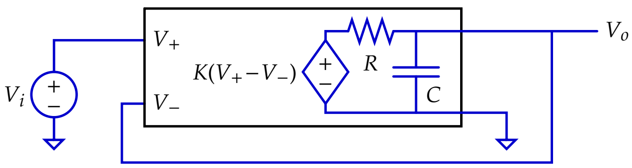
The gain K affects the speed of the system response.

Simulated responses for an op amp with $\tau = 25$ ms (typical for op amps we use in lab).



Increasing K makes the response of the system faster.

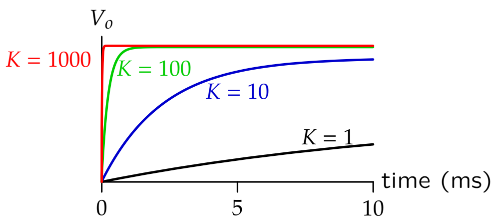
Check Yourself



Why does increasing the gain K make the response of the system faster?

Dynamic Analysis of Op Amp

Increasing K makes the system respond more quickly.



This is one of the most important uses of feedback in electronics.

Designers know how to build devices with lots of gain.

Building devices that are fast is not as easy.

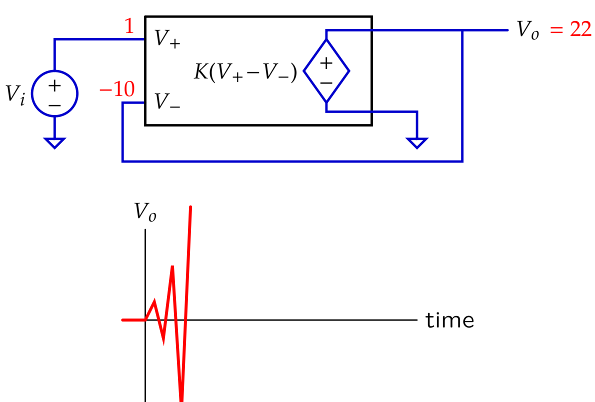
Trade gain for speed: use feedback to make circuits faster.

Summary

Today we learned how to think about dynamics of a system.

"Thinking" like an op amp

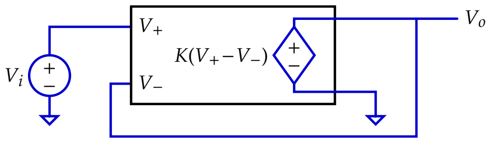
We should not think about an op amp as sequentially computing its response.



Op amp delays are more gradual!

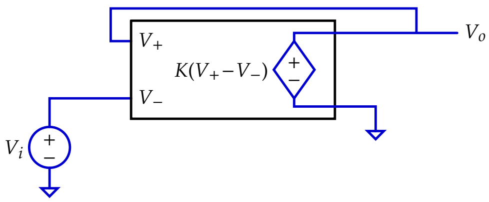
Analyzing Op Amp Circuits

Furthermore, op amps can be thought of as instantaneous constraint solvers if and only if the feedback is negative.



$$\frac{V_o}{V_i} = \frac{K}{K+1}$$

stable



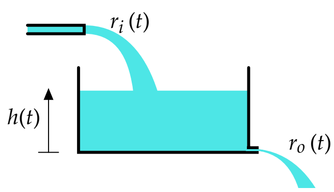
~~$\frac{V_o}{V_i} = \frac{K}{K-1}$~~

unstable

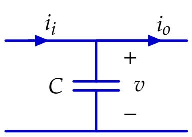
Leaky Tanks and Capacitors

Physically, an op amp operates as a pump that moves charge and thus changes the output voltage.

Water accumulates in a leaky tank and changes height.

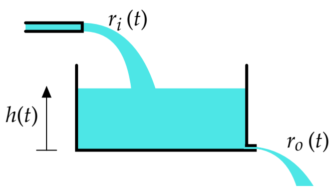


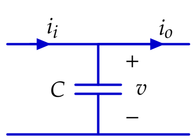
Charge accumulates in a capacitor and changes voltage.



Summary

Today we learned how to think about dynamics of a system.





We analyzed a model for the dynamics of an op amp and found a relation between gain and speed that is fundamental to the use of feedback.