MASSACHVSETTS INSTITVTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
6.01—Introduction to EECS I
Spring Semester, 2008

Week 7 Course Notes

Constraint Systems and Circuits

Circuits

Electrical circuits are made up of components, such as resistors, capacitors, inductors, and transistors, connected together by wires. You can make arbitrarily amazing, complicated devices by hooking these things up in different ways, but in order to help with analysis and design of circuits, we need a systematic way of understanding how they work.

As usual, we can't comprehend the whole thing at once: it's too hard to analyze the system at the level of individual components, so, again, we're going to build a model in terms of primitives, means of combination, and means of abstraction. The primitives will be the basic components, such as resistors and op-amps; the means of combination is wiring the primitives together into circuits. We'll find that abstraction in circuits is a bit harder than in software or linear systems: separately designed parts of a circuit tend to influence one another when they are connected together, unless you design very carefully. We'll explore a number of examples of when and how the abstractions can help us, but also when they can leave out important detail and require different models.

Constraint Models

So far, we have looked at a number of different models of systems. We have thought of software procedures as computing functions, of a robot "brain" as performing a transuction from a stream of inputs to a stream of outputs, and of linear systems as a special subclass of transductions that we can analyze for stability and other properties. In each case, we were able to construct or analyze the behavior of sub-parts of the system, as functions or transductions, and then abstract away from their implementations, use them to build more complex systems, and use the understanding of the components to understand the larger system.

Now we're going to consider a different class of systems that has a kind of modularity, but where, typically, you have to have a description of the entire system in order to say what is going to happen in a local piece of it. We will be able to view the subparts as putting "constraints" on the overall global behavior of the system; once enough pieces are put togther and their constraints are taken together, the behavior of the entire system will be specified.

One intuitive example is a set of rigid rods connected together with pins, all resting flat on a table. If we specify the x, y coordinates of the end points of one rod, and the lengths of the other rods, and the way in which they're connected together, we have described a set of constraints on the positions of all the rods. If, for example, we connect 4 rods of length 1 in a square, then the positions of the other rods are not completely specified, because the square can be squashed into a number of different rhombuses. On the other hand, if we connect only three rods into a triangle, then the position of the third vertex will be completely specified.

We will use this way of thinking about and specifying the behavior of a system to understand simple electrical circuits as systems of constraints.

Voltage and current

Voltage is a difference in electrical potential between two different points in a circuit. We will, generally speaking, pick some point in a circuit and say that it is "ground" or has voltage 0. Now, every other point has a voltage defined with respect to ground. Because voltage is a relative concept, we could pick *any* point in the circuit and call it ground, and we would still get the same results.

Current is a flow of electrical charge through a path in the circuit. A positive current in a direction is generated by negative charges (electrons) moving in the opposite direction.[1] We're not going to worry about the details of what particles are doing what (until we get to semiconductors, in another class). We'll just have to be careful when we draw and describe circuits to label the directions of the currents we're talking about.

Static circuit model

A circuit is made up of a set of components, wired togther in some structure. Each component has a current flowing through it, and a voltage difference across its two terminals (points at which it is connected into the circuit). Each type of component has some special characteristics that govern the relationship between its voltage and current.

One way to model circuits is in terms of their *dynamics*. That is, to think of the currents and voltages in the system and how they change over time. Such systems are appropriately modeled, in fact, using differential or difference equations, connected togther into complex systems, as we saw in the last couple of weeks. In the next chapter, we will consider a dynamic model of a circuit.

But for many purposes, the dynamic properties of a circuit converge quickly, and we can directly model the equilibrium state that they will converge to. The combination of the behavior of the components and the structure in which they're connected provides a set of constraints on the equilibrium state of the circuit. We'll work through this view by starting with the constraints that come from the structure, and then examining constraints for two simple types of components.

Conservation laws

The first set of constraints we get in a circuit are *conservation laws*. They describe properties of the circuit that have to be true, no matter what kinds of components we put into it. We'll describe our two conservation laws using the circuit in figure 1A. For now, don't worry about what's in the components labeled A through D. You can see that we've labeled the current through each component with an arrow, and named it i_x. We can choose these arrows to point in any direction we like, as long as we treat them consistently. For each component, we can also talk about the *voltage drop* across the component, which we've labeled v_x. It is the potential difference between

[1] At the semi-conductor level, it can also be viewed in an oversimplified way as as "holes" or positive charges moving in the direction of the current.

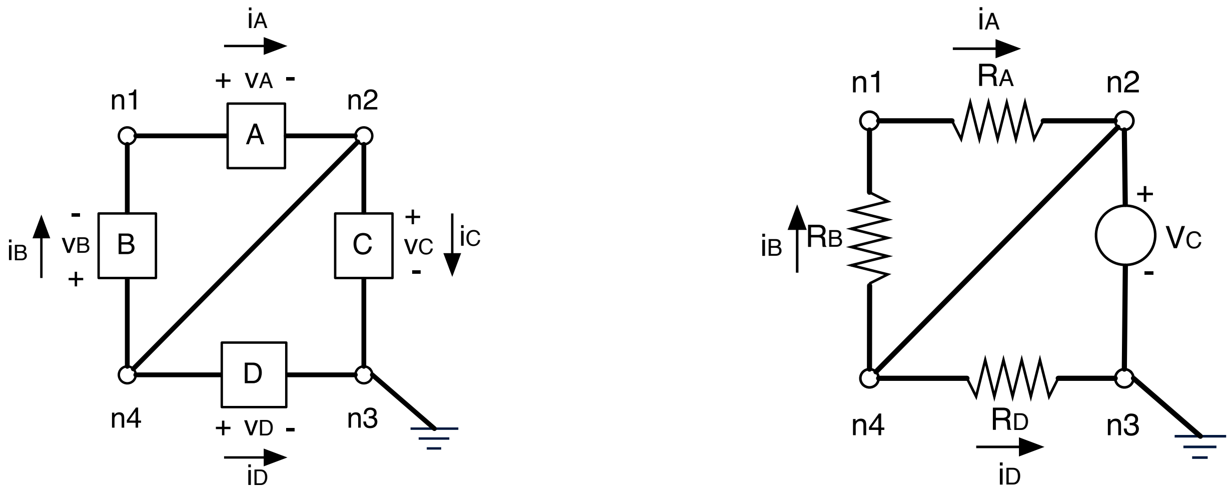


Figure 1: A. Circuit with four components. B. Circuit with three resistors and a voltage source.

the terminal labeled '+' and the terminal labeled '-', which should agree with the direction of the current for the component, flowing from '+' to '-'.

Kirchhoff's Voltage Law (KVL)

Kirchhoff's voltage law is one source of constraints that govern the behavior of a circuit. It says that:

> The algebraic sum of voltage drops taken around any loop in a network is equal to zero.

So, in our figure, we know that $-v_D + v_B + v_A + v_C = 0$. We also know that $-v_D + v_C = 0$, and that $v_B + v_A = 0$. Once you've established positive and negative terminals on your components, then be sure you as you follow a loop around, you treat the voltage drops consistently with their orientation in the circuit.

Kirchhoff's Current Law (KCL)

Each place in a circuit where two or more components connect is called a *node*, and we can label each of them with a node name.

Kirchhoff's current law is another source of constraints that govern the behavior of a circuit. It says that:

> The algebraic sum of currents entering any node must be zero.

We can write a KCL equation for each node in our circuit. Since there is a wire connecting nodes n_2 and n_4, in fact they have the same voltage, and can be considered as a single node for the purposes of analysis. So, we have, at node n_1, that $i_B - i_A = 0$. At node n_2, because it's the same as node n_4, things are a little tricky. We have incoming current from A, and current flowing out through B, C, and D. So, we get the equation: $i_A - i_B - i_C - i_D = 0$. Remember that the signs of these currents and their directions are all a matter of convention: we don't actually know yet whether the voltage at n_2 will be higher than the voltage at n_1 or not.

Node n_3 is connected to the *ground* symbol, which means we will treat it as having voltage 0. So, we can speak, now, of the voltage at node n_1, which we'll write v_1, which is really the voltage difference between n_1 and n_3. We will say that we've *solved* a circuit, when we've been able to figure out the voltages at all the nodes and the currents through all the components.

Elements

Now we need to know what the actual elements of the circuit are, in order to know how it is going to behave. In this course, we'll start by considering two very basic elements: independent voltage sources and resistors. In each case, we can describe the components in terms of a constraint they induce on the voltages and currents associated with them.

Voltage Source

An ideal voltage source with voltage v always maintains a voltage difference of v between its terminals, independent of the current flowing through the node. Batteries, in the nominal part of their operating range, can be treated as ideal voltage sources. Voltage sources are typically drawn as circles with plus and minus terminals and an associated voltage. In figure 1B, we've replaced component C with a voltage source, with voltage V_c.

Resistor

A resistor is a component that satisfies *Ohm's law*: $v = iR$, where R is the resistance, in Ohms (Ω) of the resistor, i is the current, in Amps, flowing through it, and v is the voltage drop across it, in the the same direction as the current is considered to be flowing. In figure 1B, we've replaced components A, B, and D with resistors.

Solving the circuit

Let's see if we can solve the circuit shown in figure 1B. We can write down a complete set of constraints describing the circuit, by dividing them into three groups.

KCL For every node that isn't connected to ground, assert that the sum of incoming currents is 0 (remember that nodes n_2 and n_4 are really the same):

$$\begin{aligned} i_A - i_B - i_D - i_C &= 0 \\ i_B - i_A &= 0 \ . \end{aligned}$$

Ground For every node that is connected to ground, assert that its voltage is 0:

$$v_3 = 0 \ .$$

Constitutive equations For every component, or *constituent*, in the circuit, describe the constraints it asserts on the associated voltages and currents.

$$\begin{aligned}
(v_1 - v_2) &= i_A \cdot R_A \\
(v_4 - v_1) &= i_B \cdot R_B \\
(v_4 - v_3) &= i_D \cdot R_D \\
v_2 &= v_4 \\
v_2 - v_3 &= V_c \ .
\end{aligned}$$

Because these constraints connect the components in the network structure, they will also embody the KVL constraints; it is generally much more straightforward to write down these constitutive relations than KVL constraints, and so we will proceed this way in all of our circuit analyses.

Solving So, now, if we know R_A, R_B, R_D, and V_C, which are the specifications of our components, we have 8 linear equations in 8 unknowns (v_1, v_2, v_3, v_4, i_A, i_B, i_C, and i_D). Just a small (though possibly tedious) matter of algebra, and we're done.

As an example, let $R_A = 100\Omega$, $R_B = 200\Omega$, $R_D = 100\Omega$, and $V_C = 10V$. Then, we get $v_2 = v_4 = 10V$; $i_A = i_B = 0A$ (that's reasonable: why would any current bother going that way, when it can just run through the wire from n_2 to n_4?); and $i_D = 0.1A$, which is pretty straightforward.

What happens when we take out the wire from n_2 to n_4? Now we have $i_A = i_B = i_C = -0.025A$, $i_D = 0.025A$, $v_1 = 7.5V$, $v_2 = 10V$, and $v_4 = 2.5V$.

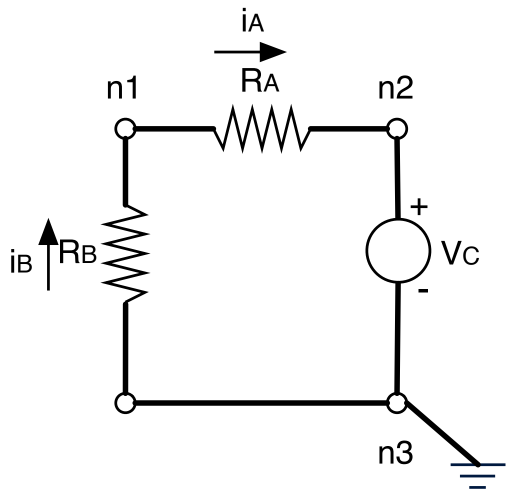
Common Patterns

There are some common patterns of resistors that are important to understand and that can be used over and over again as design elements.

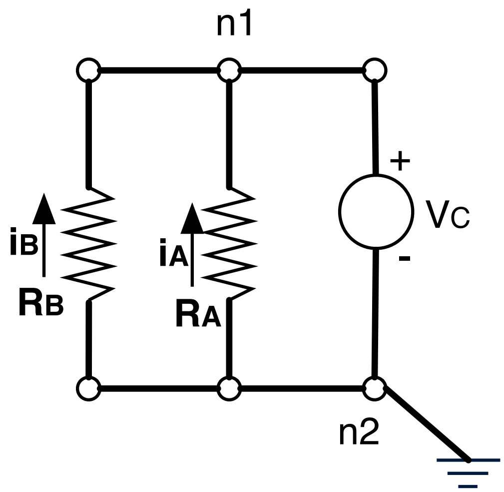
Resistors in series

Figure 2(a) shows two resistors connected together in a circuit with a voltage source. It induces a simple set of constraints:

$$\begin{aligned}
i_A - i_C &= 0 \\
i_B - i_A &= 0 \\
v_3 &= 0 \\
v_1 - v_2 &= i_A \cdot R_A \\
v_3 - v_1 &= i_B \cdot R_B \\
v_2 - v_3 &= V_c
\end{aligned}$$



(a) Series.



(b) Parallel.

Figure 2: Resistors in combination.

What happens when we solve? First, it's easy to see that because there's a single loop, KCL implies that the current across each of the nodes is the same. Let's call it i. Now, we can add together the third and fourth equations, and then use the last equation to get

$$\begin{aligned} v_3 - v_2 &= i_A R_A + i_B R_B \\ v_3 - v_2 &= i(R_A + R_B) \\ -V_c &= i(R_A + R_B) \\ -i &= \frac{V_c}{R_A + R_B} \end{aligned}$$

The interesting thing to see here is that we get exactly the same result as we would have had if there were a single resistor R, with resistance $R_A + R_B$. So, if you ever see two or more resistors in series in a circuit, with no other connections from the point between them to other components, you can treat them as if it were one resistor with the sum of the resistance values. This is a nice small piece of abstraction.

It might bother you that we got something that looks like $v = -iR$ instead of $v = iR$. Did we do something wrong? Not really. The reason that it seems funny is that the directions we picked for the currents i_A and i_B turn out to be "backwards", in the sense that, in fact, the current is running in the other direction, given the way we hooked them up to the voltage source. But the answer is still correct.

Question 1: Go back to the circuit of figure 1B with the diagonal wire removed. You should be able to construct an equivalent circuit with only one resistor. What is its resistance value?

Resistors in parallel

Now, in figure 2(b), we have a simple circuit with two resistors in parallel. Even though there are a lot of wires being connected together, there are really only two *nodes*: places where multiple components are connected. Let's write down the equations governing this system.

First, applying KCL to n_1 and n_2, we get

$$\begin{aligned} i_A + i_B - i_C &= 0 \\ -i_A - i_B + i_C &= 0 \ , \end{aligned}$$

which is the same constraint written two ways. Now, setting v_2 to ground, and describing the other components, we have:

$$\begin{aligned} v_2 &= 0 \\ v_2 - v_1 &= i_A \cdot R_A \\ v_2 - v_1 &= i_B \cdot R_B \\ v_1 - v_2 &= V_c \end{aligned}$$

We can simplify this last set of constraints to

$$\begin{aligned} -V_C &= i_A \cdot R_A \\ -V_C &= i_B \cdot R_B \end{aligned}$$

so

$$\begin{aligned} i_A &= -\frac{V_c}{R_A} \\ i_B &= -\frac{V_c}{R_B} \end{aligned}$$

Plugging these into the KCL equation, we get:

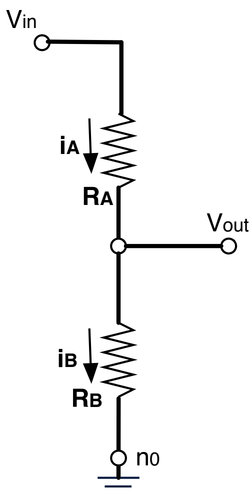
$$\begin{aligned} i_A + i_B - i_C &= 0 \\ -\frac{V_c}{R_A} - \frac{V_c}{R_B} &= i_C \\ -V_c\frac{R_A + R_B}{R_A R_B} &= i_C \\ -V_c &= i_C\frac{R_A R_B}{R_A + R_B} \end{aligned}$$

What we can see from this is that two resistances, R_A and R_B, wired up in parallel, act like a single resistor with resistance $\frac{R_A R_B}{R_A + R_B}$. This is another common pattern for both analysis and design. If you see a circuit with parallel resistors connected at nodes n_1 and n_2, you can simplify it to a circuit that replaces those two paths between n_1 and n_2 with a single one with a single resistor.

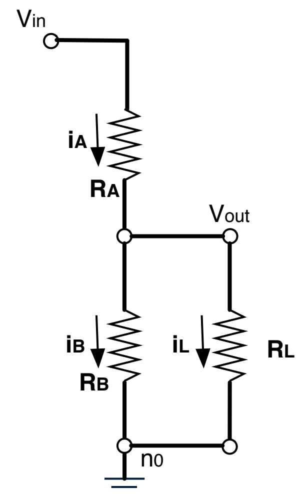
Voltage divider

Figure 3(a) shows part of a circuit, in a configuration known as a *voltage divider*. Using what we know about circuit constraints, we can determine the following relationship between V_{out} and V_{in}:

$$V_{out} = \frac{R_B}{R_A + R_B} V_{in} \ .$$



(a) Without load.



(b) With load.

Figure 3: Voltage dividers.

Let's go step by step. Here are the basic equations:

$$\begin{aligned} v_0 &= 0 \\ i_A - i_B &= 0 \\ V_{in} - V_{out} &= i_A R_A \\ V_{out} - v_0 &= i_B R_B \end{aligned}$$

We can start by seeing that $i_A = i_B$; let's just call it i. Now, we add the last two equations to each other, and do some algebra:

$$\begin{aligned} V_{in} - v_0 &= iR_A + iR_B \\ V_{in} &= i(R_A + R_B) \\ i &= \frac{V_{in}}{R_A + R_B} \\ V_{in} - V_{out} &= iR_A \\ V_{in} - V_{out} &= V_{in}\frac{R_A}{R_A + R_B} \\ V_{in}(R_A + R_B) - V_{out}(R_A + R_B) &= V_{in}R_A \\ V_{in}R_B &= V_{out}(R_A + R_B) \\ V_{out} &= V_{in}\frac{R_B}{R_A + R_B} \end{aligned}$$

So, for example, if $R_A = R_B$, then $V_{out} = V_{in}/2$. This is a very handy thing: if you need a voltage in your circuit that is between two values that you already have available, you can choose an appropriate R_A and R_B to create that voltage.

Well, almost. When we wrote $i_A - i_B = 0$, we were assuming that there was no current flowing out V_{out}. But, of course, in general, that won't be true. Consider figure 3(b). We've shown an additional "load" on the circuit at V_{out} with a resistor R_L standing for whatever resistance that additional load might offer to the ground node n_0. This changes matters considerably.

To see what is going to happen, we could solve the whole circuit again. Or, we could observe that, between the node labeled V_{out} and n_0, we have two resistances, R_B and R_L in parallel. And we've already see that resistances in parallel behave as if they are a single resistor with value $R_B R_L/(R_B + R_L)$. So, (you do the algebra), our result will be that

$$V_{out} = V_{in} \frac{R_B}{R_A + R_B + \frac{R_A R_B}{R_L}} \ .$$

The lesson here is that the modularity in circuits is not as strong as that in programs or our difference equation models of linear systems. How a circuit will behave can be highly dependent on how it is connected to other components. Still, the constraints that it exerts on the overall system remain the same.

Circuit Equivalents

We just saw that pieces of circuits cannot be abstracted as functional elements; the actual voltages and currents in them will depend on how they are connected to the rest of a larger circuit. However, we can still abstract them as sets of constraints on the values involved.

In fact, when a circuit includes only resistors and voltage sources, we can derive a much simpler circuit that induces the same constraints on currents and voltages as the original one. This is a kind of abstraction that's similar to the abstraction that we saw in linear systems: we can take a complex circuit and treat it as if it were a much simpler circuit.

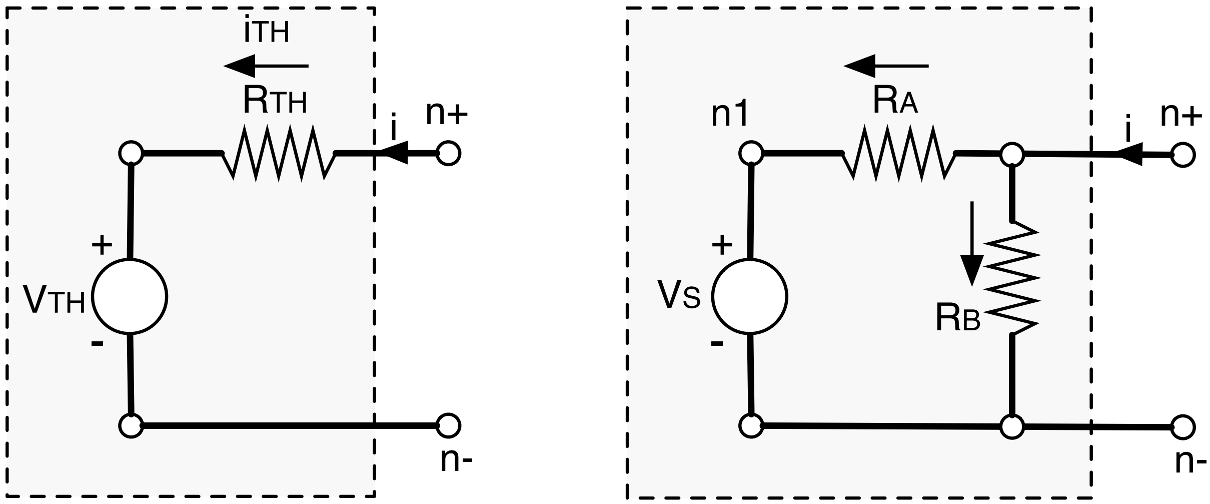
If somebody gave you a circuit made of resistors and voltage sources, and put it in a black box with two wires coming out, labeled + and -, what could you do with it? You could try to figure out what constraints that box puts on the voltage between and current through the wires coming out of the box.

We can start by figuring out the *open-current voltage* across the two terminals. That is the voltage drop we'd see across the two wires if nothing were connected to them. We'll call that V_{oc}. Another thing we could do is connect the two wires together, and see how much current runs through them; this is called the *short-circuit current*. We'll call that i_{sc}.

It turns out that these two values are sufficient to characterize the constraint that this whole box will exert on a circuit connected to it. The constraint will be a relationship between the voltage across its terminals and the current flowing through the box. We can derive it by using Thévenin's theorem:

Theorem 1 *Any combination of voltage sources and resistances with two terminals can be replaced by a single voltage source* V_{th} *and a single series resistor* R_{th}*. The value of* V_{th} *is the open circuit voltage at the terminals* V_{oc}*, and the value of* R_{th} *is* V_{th} *divided by the current with the terminals short circuited* ($-i_{sc}$)*.*

Let's look at a picture, then an example. In figure 4(a) we show a picture of a black (well, gray) box, abstracted as being made up of a circuit with a single voltage source V_{th} and a single resistor



(a) The abstract Thévenin equivalent circuit (b) An example circuit

Figure 4: Thévenin equivalence examples

R_{th} in series. The open-circuit voltage from n_+ to n_- is clearly V_{th}. The short-circuit current i_{sc} (in the direction of the arrow) is $-V_{th}/R_{th}$. So, this circuit would have the desired measured properties.[2]

Figure 4(b) shows an actual circuit. We'll compute its associated open-circuit voltage and short-circuit current, construct the associated *Thévenin equivalent* circuit, and be sure it has the same properties.

The first step is to compute the open-circuit voltage. This just means figuring out the difference between the voltage at nodes n_+ and n_-, under the assumption that the current $i = 0$. An easy way to do this is to set n_- as ground and then find the node voltage at n_+. Let's write down the equations:

$$\begin{aligned}
v_+ - v_1 &= i_A R_A \\
v_1 - v_- &= V_s \\
v_+ - v_- &= i_B R_B \\
-i_A - i_B &= 0 \\
i_A - i_S &= 0 \\
v_- &= 0
\end{aligned}$$

We can solve these pretty straightforwardly to find that

$$v_+ = V_s \frac{R_B}{R_A + R_B} \quad .$$

[2]The minus sign here can be kind of confusing. The issue is this: when we are treating this circuit as a black box with terminals n_+ and n_-, we think of the current flowing *out* of n_+ and *in* to n_-, which is consistent with the voltage difference $V_{th} = V_+ - V_-$. But when we compute the short-circuit current by wiring n_+ and n_- together, we are continuing to think of i_{sc} as flowing out of n_+, but now it is coming *out* of n_- and *in* to n_+, which is the opposite direction. So, we have to change its sign to compute R_{th}.

So, we know that, for this circuit, $R_{th} = V_s \frac{R_B}{R_A + R_B}$.

Now, we need the short-circuit current, i_{sc}. To find this, imagine a wire connecting n_+ to n_-; we want to solve for the current passing through this wire. We can use the equations we had before, but adding equation 4 wiring n_+ to n_-, and adding the current i_{sc} to the KCL equation 5.

$$\begin{aligned} v_+ - v_1 &= i_A R_A & (1) \\ v_1 - v_- &= V_s & (2) \\ v_+ - v_- &= i_B R_B & (3) \\ v_+ &= v_- & (4) \\ i_{sc} - i_A - i_B &= 0 & (5) \\ i_A - i_S &= 0 & (6) \\ v_- &= 0 & (7) \end{aligned}$$

We can solve this system to find that

$$i_{sc} = -\frac{V_s}{R_A} ,$$

and therefore that

$$\begin{aligned} R_{th} &= -\frac{V_{th}}{i_{sc}} \\ &= V_s \frac{R_B}{R_A + R_B} \frac{V_s}{R_A} \\ &= \frac{R_A R_B}{R_A + R_B} \end{aligned}$$

What can we do with this information? We could use it during circuit analysis to simplify parts of a circuit model, individually, making it easier to solve the whole system. We could also use it in design, to construct a simpler implementation of a more complex network design. One important point is that the Thévenin equivalent circuit is not exactly the same as the original one. It will exert the same constraints on the voltages and currents of a circuit that it is connected to, but will, for example, have different heat dissipation properties.

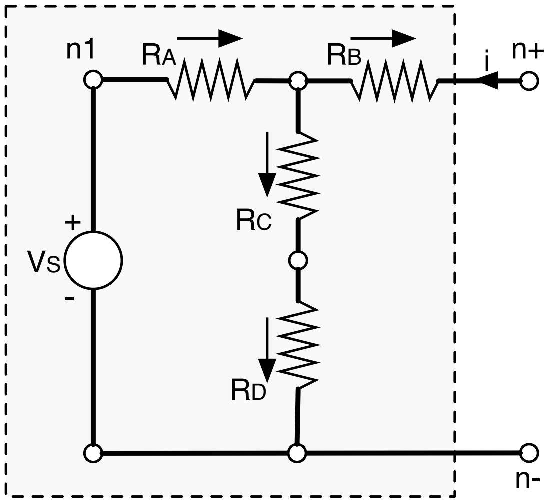
Example

Here's another example, in figure 5(a). It's a bit more hassle than the previous one, but you can write down the equations to describe the constituents and KCL constraints, as before. If we let $R_A = 2K\Omega$, $R_B = R_C = R_D = 1K\Omega$, and $V_S = 15V$, then we can solve for $V_{th} = 7.5V$ and $R_{th} = 2K\Omega$. So, it is indistinguishable by current and voltage from the circuit shown in figure 5(b).

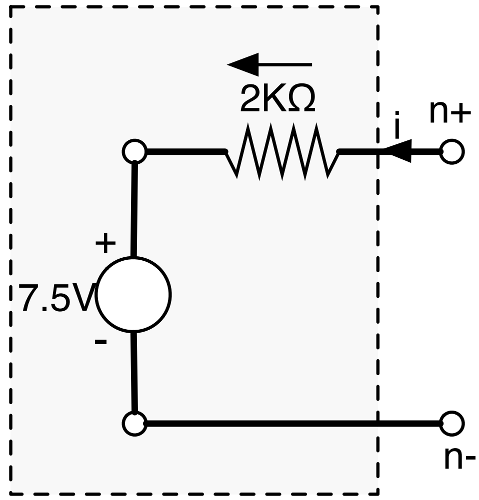
In figure 6(a) we show the same circuit, but with the connections that run outside the box made to different nodes in the circuit. Note also that the top lead is marked n_- and the bottom one n_+. If we solve, using the same values for the resistors and voltage source as before, we find that $V_{th} = -3.75V$ and $R_{th} = 1750\Omega$. We show the Thévenin equivalent circuit in figure 6(b). We've changed the polarity of the voltage source and made it 3.75V (instead of having the + terminal at the top and a voltage of -3.75), but that's just a matter of drawing.

These results are quite different: so, the moral is, it matters which wires you connect up to what!

LPK: I was inspired by the treatments in Electronic Circuits and Applications *by Wedlock and Senturia, and the Wikipedia article on voltage dividers.*

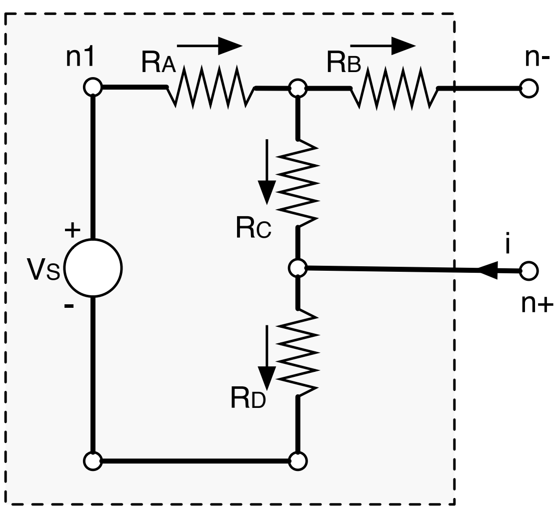


(a) Another example circuit

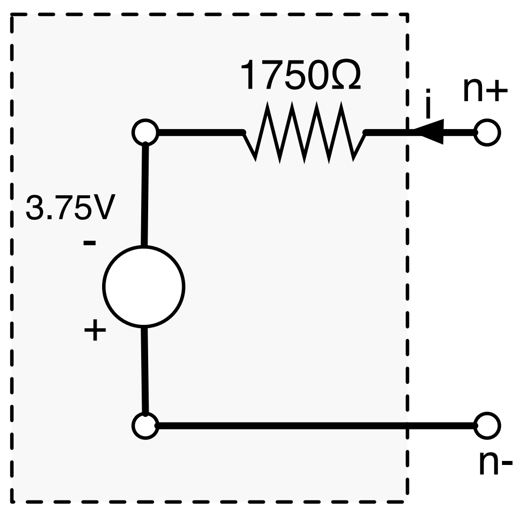


(b) Its Thévenin equivalent circuit

Figure 5: Thévenin equivalence examples



(a) Same circuit, but with different ports



(b) Its Thévenin equivalent circuit

Figure 6: Thévenin equivalence examples