MASSACHVSETTS INSTITVTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
6.01—Introduction to EECS I
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Modeling and abstraction with signals and linear systems: Part 2[1]

Note: This chapter contains questions throughout labeled "self-check." These questions are to help you check your understanding as you are reading. They are *not* part of an assignment to be written up and turned in.

We saw in the previous chapter that linear time-invariant systems can be abstracted as system functions, and remarkably, that the system function of a system—a ratio of two polynomials in $\mathcal{R}$ captures all the information about a system, namely,

Theorem:
(a) We can manipulate system functions by algebra as ratios of polynomials in $\mathcal{R}$; and
(b) if two LTI systems have the same system function then they have the same input-output behavior, i.e., for any given input, the two systems have the same output.
This is true, provided that all signals are zero before some initial n (i.e., they do not stretch backward into the indefinite past).[2]

As a consequence, we can reason about LTI systems, simply by doing algebra, and this is how we'll proceed in this chapter.

Decomposing systems as sums of modes

A first example of decomposition

The system shown in figure 1, as you can see from the top block diagram, has system function

$$H = \frac{y}{x} = \frac{1}{1 - 5\mathcal{R} + 6\mathcal{R}^2} \quad .$$

By factoring the denominator, we see that H can be rewritten in the form

$$H = \frac{1}{(1 - 2\mathcal{R})(1 - 3\mathcal{R})} \quad .$$

As we saw in the previous chapter, this shows that the poles are $p_1 = 2$ and $p_2 = 3$, and that the system can be implemented as a cascade, shown in the center image in figure 1.

[1] These notes are based on earlier drafts by Leslie Kaelbling and the treatment heres draws extensively on slides and materials prepared for 6.003 by Denny Freeman.

[2] Proof deferred to next draft of these notes.

$$H = \frac{y}{x} = \frac{1}{1 - 5\mathcal{R} + 6\mathcal{R}^2} = \frac{1}{(1 - 2\mathcal{R})(1 - 3\mathcal{R})} = \frac{-2}{1 - 2\mathcal{R}} + \frac{3}{1 - 3\mathcal{R}}$$

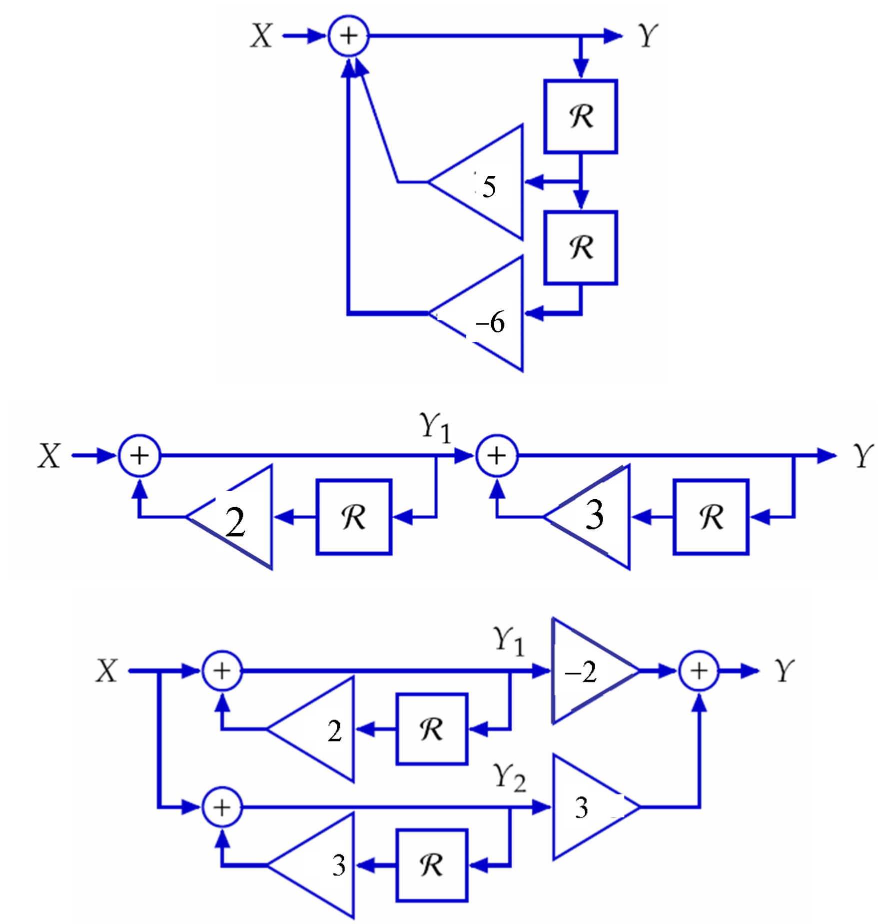

Figure 1: Manipulating the system function shows that the three systems are all equivalent.

A little more algebra reveals that H can also be written in the form

$$H = \frac{1}{(1-2\mathcal{R})(1-3\mathcal{R})} = \frac{-2}{1-2\mathcal{R}} + \frac{3}{1-3\mathcal{R}} \quad .$$

This form is particularly interesting, because it demonstrates that the system can also be implemented as a parallel combination, shown at the bottom of figure 1. In other words, we can write the system as $H = -2H_1 + 3H_2$, where

$$\begin{aligned} H_1 &= (-2) \cdot \frac{1}{1-2\mathcal{R}} \\ H_2 &= 3 \cdot \frac{1}{1-3\mathcal{R}} \quad . \end{aligned}$$

Decomposition into a sum like this provides a lot of insight into the system's behavior. As we saw at the end of the last chapter, H_1's response to the unit sample is the geometric sequence $y_1[n] = 2^n$, and H_2's response to the unit sample is $y_2[n] = 3^n$. Therefore, the entire system's response to the unit sample is

$$y[n] = -2 \cdot 2^n + 3 \cdot 3^n \quad .$$

This is a sum of two geometric series, one for each of the poles.

If we know the response to the unit sample, then we also know the response to a sum of unit samples. Take, for example, the input x with

$$x[n] = \begin{cases} a & \text{if } n = 0 \\ b & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \quad .$$

Here, x is a times the unit sample plus b times the delayed unit sample, so the system response is

$$\begin{aligned} y[n] &= a \cdot (-2 \cdot 2^n + 3 \cdot 3^n) + b \cdot \left(-2 \cdot 2^{n-1} + 3 \cdot 3^{n-1}\right) \\ &= (2a+b)(-2 \cdot 2^{n-1}) + (3a+b)(3 \cdot 3^{n-1}) \quad . \end{aligned}$$

It's still a sum of geometric series, one for each pole.

The same is true for *any* input that is a finite sum of shifted and scaled unit samples. There will be more constants lumped into the coefficients of the exponentials, but the result will still be a sum of two geometric sequences, one for each pole:

$$y[n] = c_1 \cdot 2^n + c_2 \cdot 3^n \quad .$$

So now we know the response of this system to *any* transient input x, i.e., any signal x where $x[n]$ is 0 for n outside some finite interval) is—because any such signal can be written as a finite sum of shifted and scaled unit samples. Note that the response of the system to a *transient* signal is a *persistent* signal (i.e., a signal that goes on forever). In general, a persistent response to a transient signal is called a *mode*. The modes here are sums of geometric series, one summand for each pole.

Generalizing the example: Finding the poles

There's nothing special about the poles being 2 and 3 in the the previous example. We'd get a similar result for any poles p_1 and p_2. This is because any expression of the form $1/(1-p_1\mathcal{R})(1-p_2\mathcal{R})$ has a *partial fraction decomposition*

$$\frac{1}{(1-p_1\mathcal{R})(1-p_2\mathcal{R})} = \frac{K_1}{1-p_1\mathcal{R}} + \frac{K_2}{1-p_2\mathcal{R}} \quad , \tag{1}$$

provided that $p_1 \neq p_2$, a fact you can verify by multiplying out and solving for K_1 and K_2. So the same reasoning as above tells us that the response of the system

$$H = \frac{1}{(1-p_1\mathcal{R})(1-p_2\mathcal{R})}$$

to any transient signal x is a sum of geometric sequences, one for each pole:

$$y[n] = c_1 p_1^n + c_2 p_2^n \quad . \tag{2}$$

Self-check 1: Compute the constants K_1 and K_2 in equation (1) in terms of p_1 and p_2. What goes wrong if $p_1 = p_2$?

Self-check 2: Use your answer to question (1) to work out the general case of this analysis, and equation (2) for the system

$$H = \frac{1}{(1-p_1\mathcal{R})(1-p_2\mathcal{R})}$$

($p_1 \neq p_2$) and an arbitrary transient input that is a finite sum of delayed unit samples

$$x = \sum_m^n x[j] R^j \delta \quad .$$

Give formulas for c_1 and c_2 in terms of p_1 and p_2 and the $x[j]$

Suppose we're given the system function in the form

$$H = \frac{1}{a_0 + a_1 R + a_2 R^2} \quad .$$

To find the poles, we need to factor the denominator into the form $A(1-p_1\mathcal{R})(1-p_2\mathcal{R})$. In our example above, it was easy to see by inspection that $1 - 5\mathcal{R} + 6\mathcal{R}^2 = (1-2\mathcal{R})(1-3\mathcal{R})$. In general, we reduce the the problem of finding p_1 and p_2 for which

$$a_0 + a_1 R + a_2 R^2 = A(1-p_1\mathcal{R})(1-p_1\mathcal{R}) \quad . \tag{3}$$

to the problem of finding the roots of a quadratic equation by making a substitution of variables $\mathcal{R} = 1/z$. If we do this, equation (3) becomes

$$a_0 + a_1\left(\frac{1}{z}\right) + a_2\left(\frac{1}{z}\right)^2 = A\left(1-\frac{p_1}{z}\right)\cdot\left(1-\frac{p_1}{z}\right) \quad . \tag{4}$$

Multiplying through by z^2 gives

$$a_0 z^2 + a_1 z + a_2 = A(z-p_1)(z-p_2) \tag{5}$$

which shows that the poles p_1 and p_2 are the roots of the equation

$$a_0 z^2 + a_1 z + a_2 = 0 \quad . \tag{6}$$

If we can solve the quadratic equation, we can find the poles.

Example: The Fibonacci numbers

Let's try this analysis on the *Fibonacci system*, which is given by the difference equation

$$y[n] = y[n-1] + y[n-2] + x[n] \quad .$$

If we take $x[n]$ to be the delayed unit sample, i.e., the signal where $x[1] = 1$ and $x[n] = 0$ for all other n, we get as output sequence of Fibonacci numbers, starting with $y[0] = 0$:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ \ldots \quad .$$

The system function for the Fibonacci system is

$$H = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2}$$

and so the poles are the roots of

$$z^2 - z - 1 = 0 \quad ,$$

which we can compute by the quadratic formula to be

$$(p_1, p_2) = \frac{1 \pm \sqrt{5}}{2} \quad .$$

Therefore, we have from equation (2) that the sequence of Fibonacci numbers is given by

$$y[n] = c_1 p_1^n + c_2 p_2^n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \quad .$$

We can find the values of c_1 and c_2 by using the values of $y[0]$ and $y[1]$. From $y[0] = 0$ we get

$$\begin{aligned}
&c_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^0 = 0 \\
&c_1 + c_2 = 0 \\
&c_1 = -c_2 \quad ,
\end{aligned}$$

and now using $y[1] = 1$ we get

$$\begin{aligned}
&c_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^1 = 1 \\
&c_1 \frac{1+\sqrt{5}}{2} - c_1 \frac{1-\sqrt{5}}{2} = 1 \\
&c_1 \left((1+\sqrt{5}) - (1-\sqrt{5})\right) = 2 \\
&c_1 \cdot 2\sqrt{5} = 2 \\
&c_1 = \frac{1}{\sqrt{5}} \\
&c_2 = -c_1 = -\frac{1}{\sqrt{5}}
\end{aligned}$$

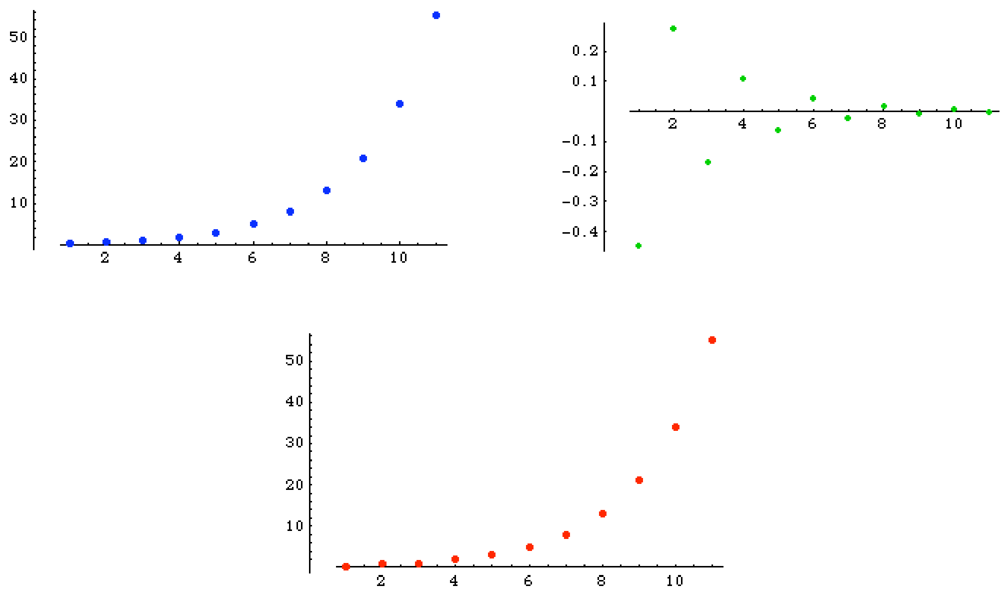

Figure 2: Plots of the two components of the Fibonacci series, and their sum. The component due to the positive root (upper left) overwhelms the other component (upper right) so the sum (bottom) is indistinguishable from the large component.

So a formula for the Fibonacci numbers is:

$$y[n] = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \quad . \tag{7}$$

It might seem remarkable that for any n, the expression in equation (7) involving powers of $\sqrt{5}$ turns out to be an integer. But it's true: the irrational number contributions from the two poles cancel each other out, leaving an integer. We'll see a similar phenomenon below with poles that are complex numbers, where the imaginary parts cancel to produce a real number.

We can relate equation (7) to the picture of qualitative behavior of single-pole LTI systems we saw in previous chapter (figure 16). We have one pole $p_1 = (1+\sqrt{5})/2 \approx 1.618$ that is positive with magnitude greater than 1. The resulting component, shown at the top left of figure 2, is a monotonically increasing exponential. The second pole $p_2 = (1-\sqrt{5})/2 \approx -0.44721$ is negative with magnitude less than one. The corresponding component, shown at the upper fight of the figure, oscillates and decays to zero. The Fibonacci series is the sum of the two components, shown at the bottom of the figure. When we form the sum, the large exponential series due to the positive root overwhelms the other one, and the graph of the sum is indistinguishable from the graph at the upper left. The fact that the second component dies out so quickly gives us an easy way to compute the Fibonacci numbers numerically: Raise 1.618 to the nth power and take the nearest integer.

Self-check 3: Find a formula for $y[n]$, where $y[n] = 2y[n-1] - 1$ with $y[0] = 0$ and $y[1] = 1$.

Complex poles

Let's examine the system given by

$$y[n] = y[n-1] - y[n-2] + x[n] \quad ,$$

which is only one minus-sign different from the equation for the Fibonacci system. Now the system function is

$$H = \frac{1}{1 - \mathcal{R} + \mathcal{R}^2}$$

and so the poles are the roots of

$$z^2 - z + 1 = 0 \quad ,$$

and applying the quadratic formula gives

$$(p_1, p_2) = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

where $j = \sqrt{-1}$. [3]

Using the same algebra as above shows that the system's response to transient signals is still given by equation (2)

$$y[n] = c_1 p_1^n + c_2 p_2^n = c_1 \left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^n + c_2 \left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)^n \quad .$$

The expression on the right-hand side, which seems *a priori* to be a complex number, will in fact be a real number. If y is the response to a sum of scaled and delayed unit samples, and the scale factors are real numbers, then c_1 and c_2 will be complex conjugates and the imaginary parts of the two summands will cancel out.

Self-check 4: Verify that c_1 and c_2 are complex conjugates, using your answer to question (2). *Hint:* Note that p_1 and p_2 are complex conjugates, as they have to be, because they are the two roots of a quadratic equation with real coefficients.

Polar representation

We can get more insight into expressions of the form $c_1 p_1^n + c_2 p_2^n$ if we express complex numbers in *polar form*. Rather than $a + bj$ we use $Me^{j\theta}$, where $M = \sqrt{a^2 + b^2}$ the *magnitude* of the number (sometimes written as $|a + bj|$) and θ is its *angle*, $\arctan(b/a)$.

If we think of (a, b) as a point in the complex plane, then M, θ is its representation in polar coordinates. This representation is justified by Euler's equation

$$e^{jx} = \cos x + j \sin x \quad ,$$

[3] Mathematicians and most scientists use i (coming from "imaginary") to denote $\sqrt{-1}$ while electrical engineers use j, because by the time engineers had figured out that complex numbers were important, i had already been used to denote electrical current. We don't know why i became the symbol for current, but people say it originated in France. Maybe someone can find out and let us know.

which can be directly derived from series expansions of e^z, $\sin z$ and $\cos z$. To see that the exponential representation is valid, we can take our number, represent it as a complex exponential, and then apply Euler's equation for complex exponentials

$$\begin{aligned} a + bj &= Me^{j\theta} \\ &= M\cos\theta + jM\sin\theta \\ &= a + bj \end{aligned}$$

Raising a complex number to a power is much more straightforwardly done in the exponential representation. In the Cartesian representation, we get big hairy polynomials. In the exponential representation, we get

$$\left(Me^{j\theta}\right)^n = M^n e^{jn\theta} \quad ,$$

which is much tidier. This is an instance of an important trick in math and engineering: changing representations. We will often find that representing something in a different way will allow us to do some kinds of manipulations more easily. There is no *one* best representation; different representations are better in different circumstances.

In the case of our particular system, we have

$$\begin{aligned} (p_1, p_2) &= Me^{\pm j\theta} \\ M^2 &= \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1 \\ \theta &= \arctan\left(\frac{1/2}{\sqrt{3}/2}\right) = \arctan\sqrt{3} = 60 \text{ degrees} \end{aligned}$$

and the output to a general transient signal is

$$y[n] = c_1 p_1^n + c_2 p_2^n = c_1 M^n e^{jn\theta} + c_2 M^n e^{-jn\theta}$$

where c_1 and c_2 are complex conjugates, $c_2 = \bar{c}_1$. Bearing in mind that $M = 1$ and that the multiples of θ are 60, 120, 180, ..., we see that the output is a repeating signal that repeats every six steps. Figure 3 (top) shows a typical response. The mode is *steady-state*: neither growing nor decaying. The particular input that generated response shown here is a unit sample plus a delayed unit sample, $\delta + \mathcal{R}\delta$.

Qualitative Behavior

We can summarize what we've seen so far:

In general, the output of a system

$$y[n] = a_0 y[n-1] + a_1 y[n-2] + x[n]$$

to any any transient input signal x will have the form given by equation (2):

$$y[n] = c_1 p_1^n + c_2 p_2^n \quad .$$

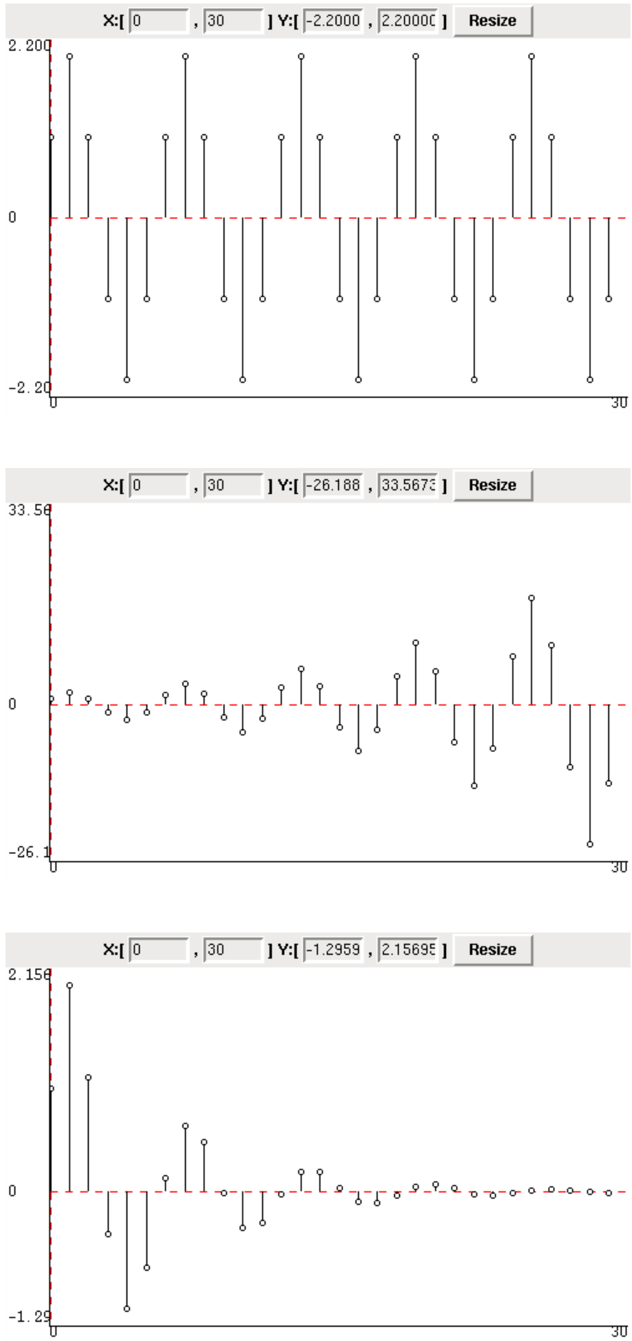

Figure 3: Plots of the typical responses of three LTI systems, showing modes that are steady-state ($M = 1$), exponentially growing ($M = 1.1$), and exponentially decaying ($M = 0.837$).

where p_1 and p_2 are the poles of the system function

$$H = \frac{1}{a_0 + a_1\mathcal{R} + a_2 R^2} \quad ,$$

provided that $p_1 \neq p_2$. The poles p_1 and p_2 are the roots of the equation

$$a_0 z^2 + a_1 z + a_2 = 0 \quad .$$

Where the roots are complex, the response can also be written in the form

$$y[n] = c_1 p_1^n + c_2 p_2^n = c_1 M^n e^{jn\theta} + c_2 M^n e^{-jn\theta}$$

where c_1 and c_2 are complex conjugates, $c_2 = \bar{c}_1$.

Let's expand this in real and imaginary parts, writing $c_1 = a + bj$, $c_2 = a - bj$, and Euler's formula $e^{jn\theta} = \cos n\theta + j \sin n\theta$. We have for $y[n]$

$$\begin{aligned}
y[n] &= (a+bj)M^n \Big[\cos n\theta + j \sin n\theta\Big] + (a-bj)M^n \Big[\cos n\theta - j \sin n\theta\Big] \\
&= M^n \Big[a\cos n\theta + aj \sin n\theta + bj \cos n\theta - b \sin n\theta + a \cos n\theta - aj \sin n\theta - bj \cos n\theta - b \sin n\theta\Big] \\
&= M^n \left[2a \cos n\theta + 2b \sin n\theta\right] \quad . \qquad (8)
\end{aligned}$$

The imaginary parts cancel out, leaving y is an exponential that grows as M^n times a sinusoid with frequency θ, that is to say, a sinusoid wiggling inside an exponential envelope. The response will be steady-state if $M = 1$, grow exponentially if $M > 1$, and decay exponentially if $M < 1$.

Figure 3 shows examples of all three behaviors. The top image, as we saw above, has $M = 1$. The middle image corresponds to

$$y[n] = 1.1y[n-1] - 1.2y[n-2] + x[n]$$

for which the poles are $(p_1, p_2) = 0.55 \pm .953j$, giving $M = 1.1$ and $\theta = 60$ degrees. The response has the same frequency as in the top image, but grows exponentially. The bottom image corresponds to

$$y[n] = 0.9y[n-1] - 0.7y[n-2] + x[n]$$

where the poles are $(p_1, p_2) = 0.45 \pm 0.705j$, so $M = 0.837$ and $\theta = 57$ degrees. The result decays exponentially and the the frequency of the oscillation is almost the same as in the other two images.

Self-check 5: Starting with the two difference equations above, compute the system poles and verify that the magnitude and frequencies are as claimed.

Self-check 6: Do a similar analysis for the system with

$$H = \frac{y}{x} = \frac{1}{1 + 0.6\mathcal{R} - 0.9R^2} \quad .$$

Do the modes grow or decay, or are they steady-state?

Response of general LTI systems to transient signals

The analysis of arbitrary LTI systems

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_k y[n-k] = b_0 x[n] + b_1 x[n-1] + \cdots + b_j x[n-j]$$

is a straightforward generalization of what we saw above. The key is that the denominator of system function

$$H = \frac{b_0 + b_1\mathcal{R} + \cdots + b_j\mathcal{R}^j}{a_0 + a_1\mathcal{R} + \cdots + a_k\mathcal{R}^k}$$

factors into a product of terms of degree one, so we can write H as

$$H = A \cdot \frac{b_0 + b_1\mathcal{R} + \cdots + b_j\mathcal{R}^j}{(1-p_1\mathcal{R})(1-p_2\mathcal{R})\cdots(1-p_k\mathcal{R})}$$

where the p_i are the poles. Just as before, we can compute the poles by making the substitution $\mathcal{R} = 1/z$ as in equations (3)–(6) to see that the poles are the roots of the polynomial equation

$$a_0 z^k + a_1 z^{k-1} + a_{k-2} z^2 + a_{k-1} z + a_k = 0 \quad .$$

So if we can find roots of polynomials, we can compute the poles. Some of the poles might be complex, but if a complex pole is present, then so is its complex conjugate. Going further, if the poles are all distinct, we can use a partial fraction expansion to express H in the form:

$$H = k \cdot \left[b_0 + b_1\mathcal{R} + \cdots + b_j\mathcal{R}^j\right] \cdot \left[\frac{C_1}{1-p_1\mathcal{R}} + \frac{C_2}{1-p_2\mathcal{R}} + \cdots + \frac{C_k}{1-p_k\mathcal{R}}\right] \quad . \tag{9}$$

How does H respond to a transient signal? We have H written in the form

$$H = H_1 \cdot H_2$$

which tells us that H is a cascade of H_1 followed by H_2.

Let's start by considering H_2. We know from the reasoning above how H_2 responds to a transient signal: The system is a parallel combination of single-pole systems, and so the response is a direct generalization of equation (8), only with more poles:

$$y[n] = c_1 p_1^n + c_2 p_2^n + \cdots + c_k p_k^n \quad . \tag{10}$$

That is, the response will be sum of modes. For the real poles we'll get increasing or decaying exponentials, depending on whether the pole absolute value of the poles is greater than or less than one. For the complex poles, each pair of complex conjugates will combine as in equation (8) to produce a sinusoid within an exponential envelope that grows or decays. In either case, the mode will grow infinitely if the magnitude of the pole is greater than 1, decay to zero if the magnitude is less than 1, or be steady-state is the magnitude is exactly 1.

Now what's the effect of cascading this system with H_1 to get the full response? The answer is that, as far as characterizing the response to transient signals goes, there's *no effect*! That's because H_1 simply a parallel combination of scaled delays: If we input a transient signal into H_1 the response will still be a transient signal that will be input to H_2, and we just above how H_2 responds to transient signals.

So putting this all together, gives us our complete result:

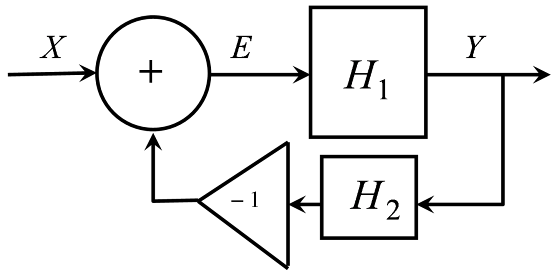

Figure 4: Feedback combination of systems H_1 and H_2.

> The response of an arbitrary LTI system to a transient signal a sum of modes corresponding to the poles. Provided that the poles are all distinct, the modes are exponentials (increasing or decaying) or exponentials times sinusoids. A key consequence is that for the system to be stable, all poles must have magnitude less than one.

In a sense, the qualitative behavior of arbitrary LTI systems in no more complicated than for the simple one-pole system—the bank account—that we saw at the beginning of chapter 1. This remarkable result is a direct reflection of the fact that the system behavior is captured by the system function. The fact that we can then do algebra on the system function to decompose it partial fractions means that the system is no more complicated than a superposition of "bank accounts."

The one gap in this analysis happens when the poles are not all distinct. In this case there's no partial fraction expansion (look again at question (1)). It turns out there are are still modes, but they aren't simple exponentials or sinusoids. You'll learn about this in more advanced courses, including 6.003.

Another thing that's missing from our analysis is how to compute the responses of LTI systems to arbitrary input signals, not just transients. It turns out that there's a beautiful theory here too, and that the response of an LTI system to a persistent sinusoid will be a sum of the modes plus a scaled and shifted copy of the input, and that scaling and shifted can be read off directly from the system function. The roots of the numerator of the system function (which are called *zeroes* of the system, play an important role here too. But that's beyond the scope of these notes.

Self-check 7: Convince yourself that the method of partial fractions that you looked at in exercise (1) for systems with two poles, really does generalize to larger numbers of poles, to yield equation (9).

Feedback and Black's Formula

Now that we have a way to determine stability of LTI systems, we can apply this to a variety of systems that are constructed using various means of composition. We've already seen two basic means of composition for LTI systems—parallel composition and cascade. Parallel composition adds the the system functions, and cascade multiplies them. A third basic means of composition is the *feedback loop*, shown in figure 4. Here, the output the output of a system H_1 is transformed by a second system H_2 and the result is subtracted from an input (often thought of as a *set point*) to yield an *error signal*, E, which is fed back into H_1.

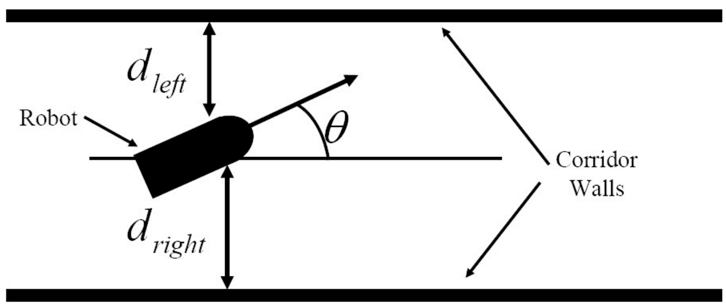

Figure 5: Robot in corridor.

We can describe this system algebraically:

$$\begin{aligned} Y &= H_1 E \\ E &= X - H_2 Y \end{aligned}$$

Solving the equations gives

$$\begin{aligned} Y &= H_1(X - H_2 Y) \\ Y + H_1 H_2 Y &= H_1 X \\ Y(1 + H_1 H_2) &= H_1 X \end{aligned}$$

which leads to

$$\frac{Y}{X} = \frac{H_1}{1 + H_1 H_2} \tag{11}$$

Equation (11) is known as *Black's formula*.[4]

Self-check 8: Show that the bank account in figure 7 of the previous chapter can be described as a feedback system. What are H_1 and H_2? Show that applying Black's formula produces the correct system function, as we've already derived in several other ways.

A robot control example

Let's apply feedback analysis to a problem you started in lab last week: controlling a robot to move down the center of a narrow corridor. The setup is shown is figure 5: The moving robot has sensors that detect its distance to the left and right walls. The figure shows have the direction of the robot's forward motion, the distance to the left wall, d_{left}, the distance to the right wall, d_{right}, and the angle of the robot with respect to the parallel walls, θ.

The goal is to design a feedback system that steers the robot down the center of the corridor while keeping it moving forward with a constant velocity.

[4]This is named after the 20th century American engineer Harold Black, who used it in his breakthrough invention of the *negative feedback amplifier*. There's a lot to say here, which we'll do in lecture and in future versions of these notes.

To get started, suppose that at step n, the robot is displaced from the center by an amount $d[n]$, and is moving with a speed V in a direction that makes an angle $\theta[n]$ as shown in the figure. Let's suppose that the step function is executed every $\delta_T$seconds. Then

$$\begin{aligned} d[n] &= d[n-1] + V\delta_T \sin\theta[n-1] \\ &\approx d[n-1] + V\delta_T\theta[n-1] \end{aligned}$$

where the approximation holds if the angle $\theta[n]$ is small. As a linear system relating output d to input θ, we have

$$\begin{aligned} d - \mathcal{R}d &= V\delta_T\theta\mathcal{R} \\ \frac{d}{\theta} &= \frac{V\delta_T\mathcal{R}}{1-\mathcal{R}} \quad . \end{aligned}$$

Let Ω be the robot's rate of turning (which is what we're going to control). The rate turning changes the angle according to

$$\theta[n] = \theta[n-1] + \delta_T\Omega[n-1] \quad .$$

As a linear system relating output θ to input Ω we have

$$\begin{aligned} \theta - \mathcal{R}\theta &= \delta_T\Omega\mathcal{R} \\ \frac{\theta}{\Omega} &= \frac{\delta_T\mathcal{R}}{1-\mathcal{R}} \quad . \end{aligned}$$

Cascading these two systems, we get a system H_{robot} that relates the distance d to the turning rate Ω:

$$\begin{aligned} H_{\text{robot}} = \frac{d}{\Omega} &= \frac{d}{\theta}\cdot\frac{\theta}{\Omega} \\ &= \frac{V\delta_T\mathcal{R}}{1-\mathcal{R}}\cdot\frac{\delta_T\mathcal{R}}{1-\mathcal{R}} \\ &= \frac{(\delta_T)^2 V\mathcal{R}^2}{(1-\mathcal{R})^2} \quad . \end{aligned} \tag{12}$$

Now for the control: Let's suppose that we want to keep the robot at a distance $D_{\text{desired}}[n]$ from the center of the corridor at each step n. For the problem as we specified it, $D_{\text{desired}}[n]$ would be a zero for all n, but we could imagine other possibilities. Let's set $E[n] = D_{\text{desired}}[n] - d[n]$, the "error," to be the difference between where we want the robot to be, and where it actually is.

Suppose now, that we have a control law that relates the rate of turning Ω to the error E according to some system function H_{control}:

$$H_{\text{control}} = \frac{\Omega}{E} \quad . \tag{13}$$

For example, we might want to make Ω be simply proportional to E, or try some more elaborate scheme.

Putting this all together, we have the feedback control system shown in figure 6. The output d from H_{robot} is fed back and subtracted from the desired value, to produce an error that, when transformed by H_{control}, becomes the input to H_{robot}.

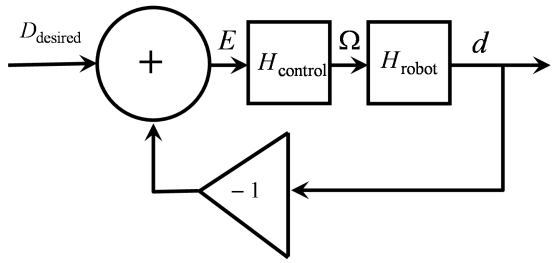

Figure 6: The robot control system for following a corridor.

This is precisely the setup for Black's formula, which lets us compute the system function of overall feedback system as

$$\frac{H}{1+H}$$

where $H = H_{\text{control}} \cdot H_{\text{robot}}$. Thus we have

$$\frac{d}{D_{\text{desired}}} = \frac{H_{\text{control}}H_{\text{robot}}}{1 + H_{\text{control}}H_{\text{robot}}} \quad . \tag{14}$$

Given any particular control law H_{control} we can investigate the stability of the control system by using equations (12) and (14) to compute the poles.

Self-check 9: Suppose we make the robot's rate of turning proportional to $E = D_{\text{desired}} - d$, i.e., we take as our control law $H_{\text{control}} = kE$. Compute the poles of the system as a function of k. For which values of k is the system stable?

Self-check 10: Carry out the same analysis with the control law

$$\Omega[n] = k_1 E[n] + k_2 E[n-1] \quad .$$

Compute the poles as k_1 and k_2 vary and analyze the stability of the system.

Self-check 11: Write a computer program to compute the poles of the feedback system with the control law in question (10) for various values of k_1 and k_2. Can you find ranges for which the control law produces a stable system?