Linear time-invariant systems (LTI)

- *As block diagrams:* Build from adders, (constant) gains, and delays – no other elements
- *As difference equations:* Linear with constant coefficients

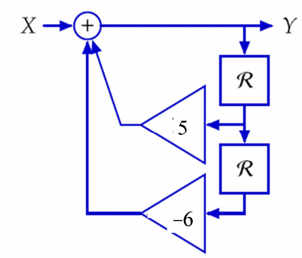
$$a_0 y[n] + a_1 y[n-1] + \cdots a_k y[n-k]$$
$$= \; b_0 x[n] + b_1 x[n-1] + \cdots b_j x[n-j]$$

- *As operator equations:* Built from addition, multiplication by constants, and "multiplication" by R

$$a_0 y + a_1 R y + \cdots a_k R^k y$$
$$= \; b_0 x + b_1 R x + \cdots b_j R^j x$$

Quiz (review)

- What is the system function for



The big fact

- The operator algebra mirrors the behavior of the system, so we can reason about combining systems by doing algebra.
- This is captured by the idea of a **system function**

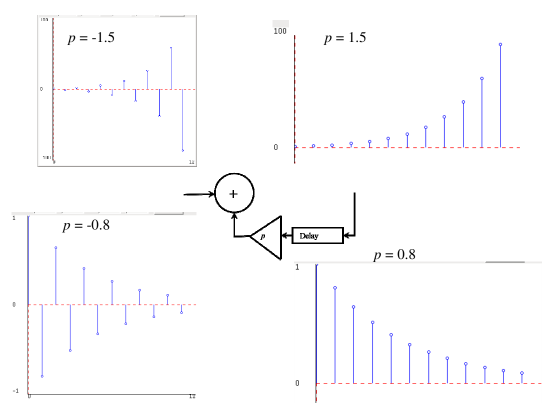
$$H = \frac{\text{output}}{\text{input}} = \frac{y}{x} = \frac{b_0 + b_1 R + \cdots b_j R^j}{a_0 + a_1 R + \cdots a_k R^k}$$

Theorem

- We can manipulate operator expressions with the rules of ordinary algebra, including multiplication by R.
- If two LTI systems have the same system function, then they have the same input/output behavior
- *Provided:* All input signals are zero before some initial time

PCAP framework for signals and systems

Primitives	signal
Combination	adder, gain, delay
Abstraction	system function
Patterns	



Poles

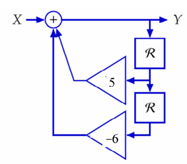
- The system function can be written in the form

$$\frac{\text{polynomial in } R}{(1-p_1R)(1-p_2R)\cdots(1-p_kR)}$$

- The p's are the poles

Quiz

What are the poles for



$$H=\frac{1}{1-5R+6R^2}$$

Partial fraction expansion

Any fraction
$$\frac{1}{(1-p_1R)(1-p_2R)\cdots(1-p_kR)}$$

Can be written in the form

$$\frac{c_1}{(1-p_1R)}+\frac{c_2}{(1-p_2R)}+\cdots+\frac{c_k}{(1-p_kR)}$$

Provided the p's are all distinct

Proof by algebra: Clear the fractions and solve for the c's

How to compute the poles

Problem: Find the p's such that

$$a_0 + a_1 R + a_2 R^2 + \cdots + a_k R^k = A(1 - p_1 R)(1 - p_2 R)\cdots(1 - p_k R)$$

Solution: Substitute R=1/z to get

$$a_0 z^k + a_1 z^{k-1} + a_2 z^{k-2} + \cdots + a_k = A(z - p_1)(z - p_2)\cdots(z - p_k)$$

So the p's are the roots of

$$a_0 z^k + a_1 z^{k-1} + a_2 z^{k-2} + \cdots + a_k = 0$$

Fibonacci system

$$y[n] = y[n-1] + y[n-2] + x[n]$$

$$H = \frac{y}{x} = \frac{1}{1 - R - R^2}$$

Poles of Fibonacci system

$$H = \frac{y}{x} = \frac{1}{1 - R - R^2}$$

The poles are the roots of

$$z^2 - z - 1 = 0$$

$$(p_1, p_2) = \frac{1 \pm \sqrt{5}}{2}$$

$p_1 = 1.618 \qquad p_2 = -0.4471$

Formula for the Fibonacci numbers

$$y[n] = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Take y[0]=0 and y[1]=1 and solve for c1 and c2:

$$y[n] = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

This is an integer for all n!

Approximating fib[n]

fib[n]: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55,

```
>>> phi=(1+sqrt(5))/2
>>> def fibApprox(n):
        return phi**n/sqrt(5)

>>> fibApprox(4)
3.0652475842498532
>>> fibApprox(8)
21.009519494249016
>>> fibApprox(10)
55.003636123247432
>>> fibApprox(20)
6765.0000295639356
>>>
```

Quiz

Find the poles:

$$y[n] = y[n-1] - y[n-2] + x[n]$$

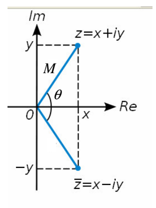
$$H = \frac{1}{1 - R + R^2}$$

Complex numbers in polar form

$z = x + yj = Me^{j\theta}$

$M = \sqrt{x^2 + y^2} \quad \theta = \arctan(y/x)$

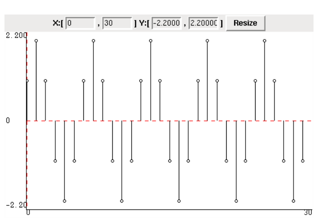
$z^n = M^n e^{jn\theta}$



$$\begin{aligned} y[n] &= (a+bj)M^n e^{nj\theta} + (a-bj)M^n e^{-nj\theta} \\ &= 2M^n(a\cos n\theta + b\sin n\theta) \end{aligned}$$

This is an exponential that gows or decays as M^n, times a sinusoid of frequency θ

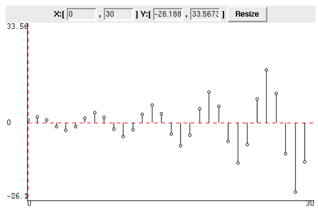
See the course notes for the computation



$y[n] = y[n-1] - y[n-2] + x[n]$

$p = 0.5 \pm j\frac{\sqrt{3}}{2}$

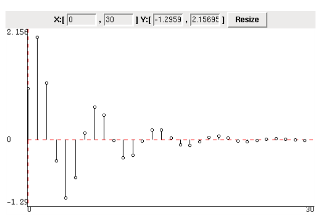
$M = 1 \quad \theta = 60^\circ$



$y[n] = 1.1\, y[n-1] - 1.2\, y[n-2] + x[n]$

$p = 0.55 \pm 0.953\, j$

$M = 1.1 \quad \theta = 60^\circ$



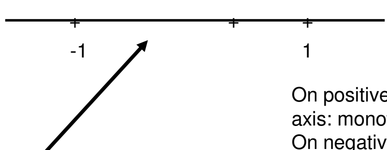
$y[n] = 0.9\, y[n-1] - 0.7\, y[n-2] + x[n]$

$p = 0.45 \pm 0.705\, j$

$M = 0.837 \quad \theta = 57^\circ$

Stability for “bank account” system

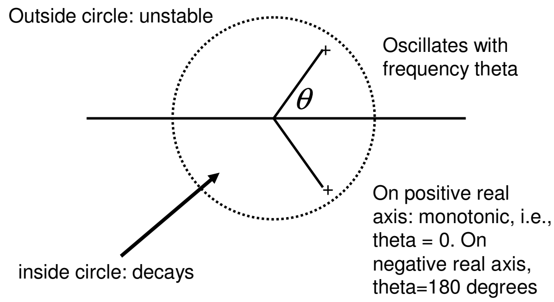
Outside interval [-1,1]:
unstable



On positive real axis: monotonic, i.e.,
On negative real axis, oscillates

inside interval: decays

Stability in general, with complex poles



Generalizing the story (I): To arbitrary transient inputs

- The response to a sum of impulses is a sum of the responses. So we just add them up. The answer is still of the form a sum of constants times powers of the poles.

Generalizing the story (II): To more that two poles

- The partial fraction decomposition still works (assuming the poles as all distinct). So the result is still a sum of powers of the poles.
- The complex poles come in conjugate pairs, so they combine to form sinusoids.
- So everything is just like before, except there are more terms added in.

Generalizing the story (III): To arbitrary system functions

For

$$H = \frac{1}{a_0 + a_1 R + \cdots a_k R^k}$$

We know the form of the response to transient signals. What can we say about

$$H = \frac{b_0 + b_1 R + \cdots b_j R^j}{a_0 + a_1 R + \cdots a_k R^k}$$

Answer

- Having the numerator there doesn't change the general form of the response (although the constants change).

The final result

- For any LTI system, the response to any transient signal is a sum of geometric series in the poles*

$$c_1 p_1^n + c_2 p_2^n + \cdots c_k p_k^n$$

* provided the poles are distinct

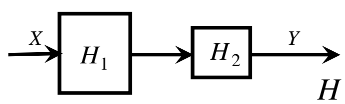
- If any pole has magnitude > 1, the system is unstable.
- For complex poles, the conjugate pairs combine to give components that oscillate with frequency determined by the angle of the pole

Generalizing the story (more)

- When poles not distinct
- To arbitrary inputs
- To signals that originate in the indefinite past (audio processing)
- To two dimensional signals (image processing)
- To continuous as well as discrete systems

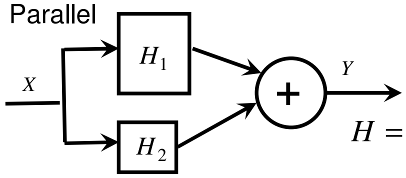
Means of combination for systems

Cascade



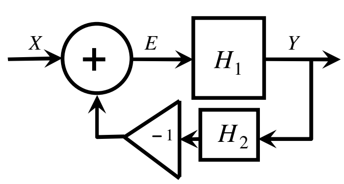
$$H = H_2 H_1$$

Parallel



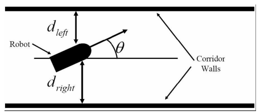
$$H = H_1 + H_2$$

Feedback combination



Black's formula $$H = \frac{H_1}{1 + H_1 H_2}$$

Robot in corridor

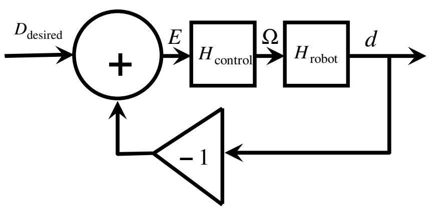


$$E = D_{\text{desired}} - d$$

$$d[n] = d[n-1] + \delta t V \theta[n-1] \qquad \frac{d}{\theta} = \frac{\delta t V R}{1-R}$$

$$\theta[n] = \theta[n-1] + \delta t[\Omega - 1] \qquad \frac{\theta}{\Omega} = \frac{\delta t R}{1-R}$$

$$H_{\text{robot}} = \frac{d}{\Omega} = \frac{(\delta t)^2 V R^2}{(1-R)^2} \qquad H_{\text{control}} = \frac{\Omega}{E}$$



$$\frac{d}{D_{\text{desired}}} = \frac{H_{\text{control}} H_{\text{robot}}}{1 + H_{\text{control}} H_{\text{robot}}}$$

Last week you used $H_{\text{control}} = k$