Interpreting Difference Equations and System Functions

Okay, so we've got a difference equation that describes our system. We can turn it into a transfer function, and back again if we wanted to. But what's so cool about the transfer function anyway? Sure, transfer functions allow us to use algebra to combine systems in difference equation or block diagram form, but there's more to it. The transfer function can give us insight into the behavior of the system.

Finding the Poles

So, we've got the transfer function of our system of interest. For the purposes of 6.01, we'll only examine the denominator of the function.[1] Our denominator is a polynomial in R. In the case of a first order denominator, we can manipulate the transfer function to be of the form

$$H=\frac{A}{1-\mathrm{p}R}$$

where A is the numerator. The factor *p* is what we care about here: it's the pole of our system. In the one-pole system case, the pole tells us a lot about the qualitative nature of the system. Its magnitude tells us about the stability of the system: a pole of magnitude greater than one leads to an *unstable* system, while a pole with magnitude less than one will lead to a *stable* system. The sign of our pole tells us about the system behavior. A positive pole leads to monotonic behavior, while a negative pole will produce an oscillatory response. See pg 20 of the Course Notes from week 5 for graphs of each behavior.

Multi-Pole Systems

Okay, we know what to do if our system only has one pole. But what if the denominator of our transfer function is a higher order polynomial? Say we have something of the form:

$$H=\frac{B}{a_0+a_1 R+\cdots a_k R^k}$$

Again, we have some arbitrary numerator B. If we're awesome at algebra, we can factor that mess into:

$$A(1-p_1 R)(1-p_2 R)\cdots(1-p_k R)$$

Each of the indexed p's is a pole in the system, and we will analyze the system in much the same way. But, chances are that factoring the denominator in that manner is difficult. Thus, we will perform a substitution to make our lives easier. We'll just replace all the R's with 1/z, multiply through by z raised to the power of the order of the polynomial, and then find the roots of that equation. It turns out that those roots are the poles we're looking for. An algebraic demonstration is in order. Starting with:

$$H=\frac{1}{1-5R+6R^2}$$

$$H=\frac{1}{1-5z^{-1}+6z^{-2}}$$

$$H=\frac{z^2}{z^2-5z+6}$$

Taking the roots of the denominator, we find that our poles are 2 and 3.

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1 In more advanced courses, we will also analyze the numerator of the transfer function. By performing the same analysis on the numerator that we use on the denominator to find the poles, we find the *zeroes* of the transfer function.

Behavior of Multi-pole systems; Complex Poles

For the purposes of stability, all the poles need to have magnitude less than one to ensure a stable system that does not "blow up". That's an easy rule. But what happens if the poles turn out to be complex numbers? Since we've been looking at the magnitudes of the poles, nothing changes on the stability front. The response will turn out to oscillate, just as it did in for a negative pole in the single-pole case. See pg 7-8 of the Week 6 Course notes for some insight as to why that is the case.

Responses of Systems to Transient Signals; Initial Conditions

It turns out that the poles describe how a system will behave when given a transient input (an input that goes away after some time *t*). The system response to a transient signal will be a sum of geometric sequences. Each pole[2] will provide one such geometric sequence, in the form:

$$y[n]=c_1 p_1^n+c_2 p_2^n+\cdots+c_k p_k^n$$

The path we took to get here is best covered by Weeks 5 & 6 of the course notes. From that equation, we can see why the pole magnitude is our stability criterion: the transient response of poles of magnitude less than one is an exponential decay, while the transient response of poles of magnitude greater than one is an exponential growth.

How do we find the values of the constant coefficients? We plug in the initial conditions. For a difference equation, this means we need one initial condition for every delay in our system. In the review session, I'll work the Fibonacci Number example on pg 5 of the Week 6 notes.

Because we're dealing with linear systems, we can take each term of the system response (called a *mode* of the system) and analyze it separately. The total system response is just the sum of the response to each mode. Thus, if one of the modes of a system is unstable, the whole system is unstable.

Summary of Mode Behavior as Determined by Poles

Behavior of mode	Pole a positive real number	Pole negative or complex
Pole Magnitude > 1	Unstable, Monotonic response	Unstable, Oscillatory response
Pole Magnitude < 1	Stable, Monotonic response	Stable, Oscillatory response

2 This analysis will break down in the case of repeated (double, triple, etc) roots, where we will need to do something slightly more clever. Such methods are beyond the scope of this course.