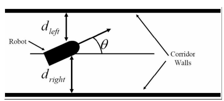
Review example 1: Aunt Zelda's bank account growing from some initial balance

$$y[n+1] = y[n] + .05\,y[n]$$
$$y[n+1] - 1.05\,y[n] = 0$$

Natural frequency is 1.05

solution grows as 1.05^n

Review example 2: Your robot from lab



$d = d_{right} - d_{left}$ angular velocity = gain $\times d = -Kd$

$d[n+1] - d[n] - \delta t V \theta[n] = 0$

$\theta[n+1] - \theta[n] - \delta t K d[n] = 0$

…after some algebra…

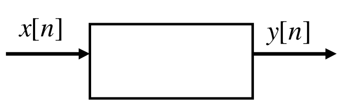
$d[n+2] - 2d[n+1] + \left(1 - (\delta t)^2 VK\right) d[n] = 0$

natural frequencies were complex, outside unit circle

so system oscillates unstably

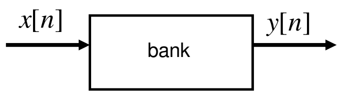
From difference equations to input-output systems

- Rather than a single difference equation, think about transforming a sequence of inputs x[n] to a sequence of outputs y[n].
- This models lots of situations



Example 1: Aunt Zelda redux

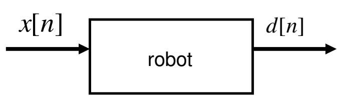
- Aunt Zelda doesn't die, but continues to make deposits and withdrawals:



$$y[n+1]-1.05y[n]=x[n]$$

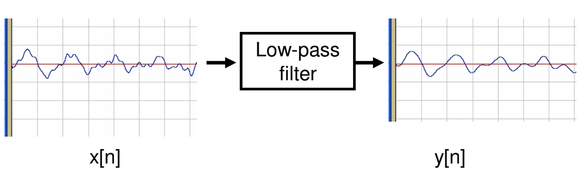
Example 2: A more general robot control program

- Rather than taking angular velocity x, to be Kd, let it be set by your program in some other way



$$d[n+1]-d[n]-\delta t V\theta[n]=0$$
$$\theta[n+1]-\theta[n]-\delta t\ x[n]=0$$
$$x[n]=\text{Your control algorithm}\{\text{measurements}\}$$

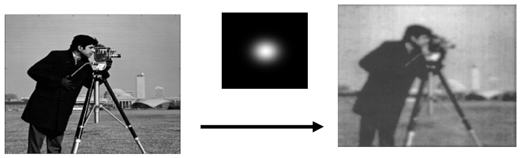
Example 3: Sound processing



$$y[n]=b_{-2}x[n-2]+b_{-1}x[n-1]+b_0x[n]+b_1x[n+1]+b_2x[n+2]$$

= weighted average of 5 points around x[n]

Example 4: 2D image processing



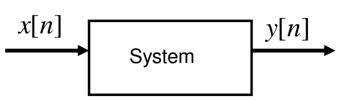
Nowak, R. (2005, July 22). Image Restoration Basics. Retrieved from the Connexions Web site: http://cnx.org/content/m10972/2.2/

Linear shift-invariant systems

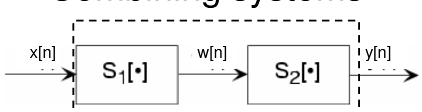
- General form of the transformations we're dealing with

$$\sum_{k=0}^{M} a_k\, y[n+k] = \sum_{l=0}^{N} b_l\, x[n+l]$$

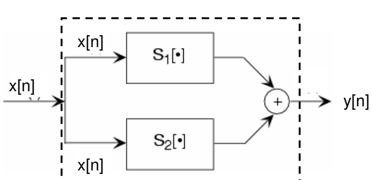
Think of this as a way of transforming one sequence into another



Combining systems



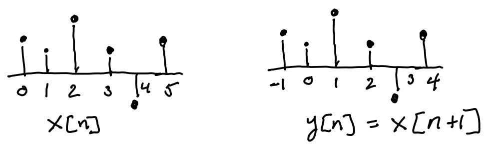
composition (cascade)



parallel sum

Combining sequences

- Addition: $y[n] = x_1[n] + x_2[n]$
- Scaling: $y[n] = k\,x[n]$
- Shift: $y[n] = x[n+1]$



Framework for abstraction

	sequences	systems
primitives		
Means of combination	addition scaling shift	cascade parallel sum
Means of abstraction	How do we make combinations easier to work with?	
Means of capturing common patterns		

The big idea

- Invent a way to model sequences and systems in terms of something that models the means of combination as ordinary algebra
- This lets us analyze systems using ordinary algebra
- A method for doing this is called the *Z-transform*

The Z-transform of a sequence x[n]

- Let the x[n] be the coefficients of a power series in a variable called *z*.
- The resulting function of *z* is the Z-transform, $\tilde{X}(z)$ written

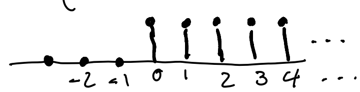
$$\tilde{X}(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Note that $x[n]$ is the coefficient of z^{-n}

This is the bilateral Z-transform

Example 2

$$x[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



$$\begin{aligned} \tilde{X}(z) &= 1 \cdot z^0 + 1 \cdot z^{-1} + 1 \cdot z^{-2} + 1 \cdot z^{-3} + \cdots \\ &= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots \\ &= \frac{1}{1 - z^{-1}} \end{aligned}$$

Example 3

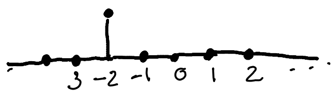
$$x[n] = \begin{cases} \alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\tilde{X}(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1 - \alpha z^{-1}}$$

Quiz

- What is the Z-transform of



Why the Z-transform is nice

- addition and scaling transform to addition and scaling

$$x[n] \longleftrightarrow \tilde{X}(z)$$

$$x_1[n] + x_2[n] \longleftrightarrow \tilde{X}_1(z) + \tilde{X}_2(z)$$

$$k\,x[n] \longleftrightarrow k\,\tilde{X}(z)$$

Why the Z-transform rocks

- Shifting transforms to multiplication by z:

If $y[n] = x[n+1]$

then $\tilde{Y}(z) = z\,\tilde{X}(z)$

Proof

If $y[n] = x[n+1]$ then

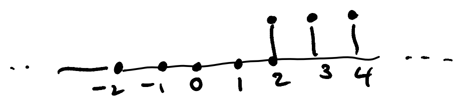
$$\tilde{Y}(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n+1] z^{-n}$$

$$= z \sum_{n=-\infty}^{\infty} x[n+1] z^{-(n+1)}$$

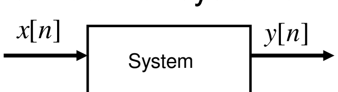
$$= z \sum_{n=-\infty}^{\infty} x[n] z^{-n} = z \tilde{X}(z)$$

Quiz

- What is the Z-transform of



What does this mean for systems?



$$\sum_{k=0}^{M} a_k y[n+k] = \sum_{l=0}^{N} b_l x[n+l]$$

For the transforms, we have

$$\left(\sum a_k z^k\right)\tilde{Y}(z) = \left(\sum b_l z^l\right)\tilde{X}(z)$$

$$\frac{\tilde{Y}(z)}{\tilde{X}(z)} = \frac{\sum b_k z^k}{\sum a_l z^l} = \tilde{H}(z)$$

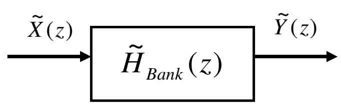
Welcome to the frequency domain

$$\tilde{Y}(z) = \tilde{H}(z)\tilde{X}(z)$$

$\tilde{H}(z)$ is called the system function

- The system function is a rational function (ratio of polynomials)
- The poles of the system function (roots of the denominator) determine whether the system is stable

Aunt Zelda's bank



$$y[n+1] - 1.05\,y[n] = x[n]$$

$$z\tilde{Y}(z) - 1.05\tilde{Y}(z) = \tilde{X}(z)$$

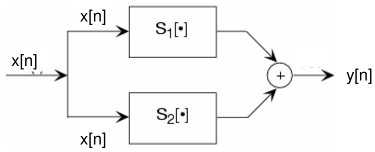
$$\tilde{H}_{Bank}(z) = \frac{\tilde{Y}(z)}{\tilde{X}(z)} = \frac{1}{z - 1.05}$$

Quiz

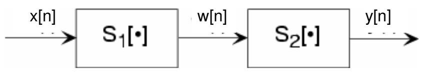
$$y[n+2] - 3y[n+1] + y[n] = 2x[n+1] + 3x[n]$$

What's the system function?

What happens with system functions when we combine systems?

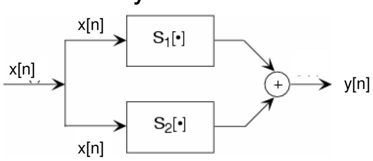


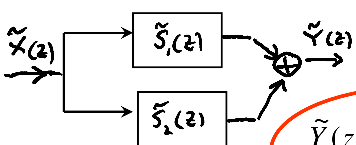
parallel sum



composition (cascade)

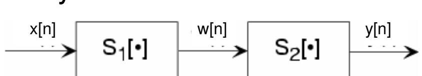
System function of a sum

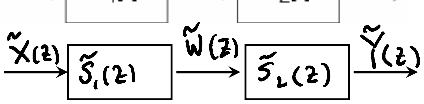




$$\frac{\tilde{Y}(z)}{\tilde{X}(z)} = \tilde{S}_1(z) + \tilde{S}_2(z)$$

System function of a cascade



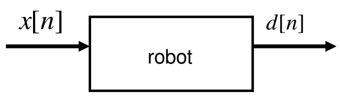


$$\tilde{Y}(z) = \tilde{S}_2(z)\tilde{W}(z) = \tilde{S}_2(z)\left(\tilde{S}_1(z)\tilde{X}(z)\right)$$
$$= \left(\tilde{S}_2(z)\tilde{S}_1(z)\right)\tilde{X}(z)$$

$$\frac{\tilde{Y}(z)}{\tilde{X}(z)} = \tilde{S}_2(z)\tilde{S}_1(z)$$

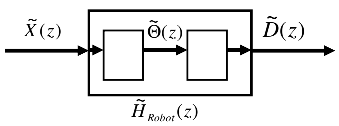
Example 2: A more general robot control program

- Rather than taking angular velocity x, to be Kd, let it be set by your program in some other way



$$d[n+1]-d[n]-\delta t V\theta[n]=0$$
$$\theta[n+1]-\theta[n]-\delta t\ x[n]=0$$
$$x[n]=\text{Your control algorithm}\{\text{measurements}\}$$

Example: Analyzing the robot control program in the frequency domain



$d[n+1]-d[n]=\delta t V\theta[n]$ $\qquad \tilde{D}(z)=\dfrac{\delta t V}{z-1}\tilde{\Theta}(z)$

$\theta[n+1]-\theta[n]=\delta t\ x[n]$ $\qquad \tilde{\Theta}(z)=\dfrac{\delta t}{z-1}\tilde{X}(z)$

$$\tilde{D}(z)=\frac{\delta t V}{z-1}\cdot\frac{\delta t}{z-1}\tilde{X}(z)=\frac{(\delta t)^2 V}{z^2-2z+1}\tilde{X}(z)$$

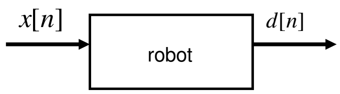
$$d[n+2]-2d[n+1]+d[n]=(\delta t)^2 V x[n]$$

Framework for abstraction

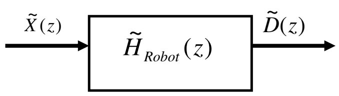
	sequences	systems
primitives		
Means of combination	addition scaling shift	cascade parallel sum
Means of abstraction	Z-transform	System function
Means of capturing common patterns		

Example 2 (continued)

- Let d[n] be the difference d_{right}[n]-d_{left}[n]. Let x[n] be the robot's angular velocity



$$d[n+1]-d[n]-\delta t V\theta[n]=0$$
$$\theta[n+1]-\theta[n]-\delta t\ x[n]=0$$
$$\boxed{x[n]=\text{Your control algorithm}\{\text{measurements}\}}$$



$$\tilde{H}_{Robot}(z)=\frac{\tilde{D}(z)}{\tilde{X}(z)}=\frac{(\delta t)^2 V}{z^2-2z+1}$$

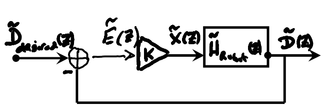
Here's the method you used in lab last week

$$x[n]=K\,e[n]=K\left(d_{desired}[n]-d[n]\right)$$

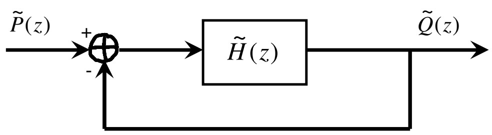
I.e., set x[n] to be some constant K times the error, where the error is the difference between what we want and what we have.

Let's use frequency-domain methods to redo the same analysis we did previously.

Block diagram for complete robot control system



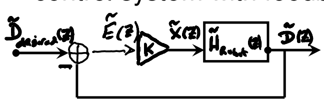
Negative feedback configuration



$$\frac{\tilde{Q}(z)}{\tilde{P}(z)} = \frac{\tilde{H}(z)}{1+\tilde{H}(z)}$$

Black's formula

Compute system function for overall robot control system with feedback



$$\frac{\tilde{D}(z)}{\tilde{D}_{desired}(z)} = \frac{\tilde{H}(z)}{1+\tilde{H}(z)}$$

where

$$\tilde{H}(z) = K\tilde{H}_{Robot}(z) = \frac{K(\delta t)^2 V}{z^2 - 2z + 1}$$

Compute system function for overall robot control system with feedback (cont)

$$\frac{\tilde{D}(z)}{\tilde{D}_{desired}(z)} = \frac{\frac{K(\delta t)^2 V}{z^2 - 2z + 1}}{1 + \frac{K(\delta t)^2 V}{z^2 - 2z + 1}}$$

$$= \frac{K(\delta t)^2 V}{z^2 - 2z + \left(1 + K(\delta t)^2 V\right)}$$

Compute system function for overall robot control system with feedback (cont)

$$\frac{\tilde{D}(z)}{\tilde{D}_{desired}(z)} = \frac{\dfrac{K(\delta t)^2 V}{z^2 - 2z + 1}}{1 + \dfrac{K(\delta t)^2 V}{z^2 - 2z + 1}}$$

$$= \frac{K(\delta t)^2 V}{z^2 - 2z + \left(1 + K(\delta t)^2 V\right)}$$

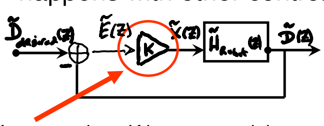
$$1 \pm \sqrt{-K(\delta t)^2 V}$$

Natural frequencies are the roots of the denominator

This is unstable, just as we knew at the beginning of the lecture.

So what's the point of going through this?

Now we have a way of analyzing what happens with other control laws



We can replace K by a more elaborate control law, for example

$$x[n] = K_1 e[n] + K_2 e[n-1]$$

Redo the analysis with

$$\frac{\tilde{X}(z)}{\tilde{E}(z)} = K_1 + K_2 z^{-1}$$

END