6.081 10/24/06

Quick Reminder about Homogenous Difference Equations

Normal Form

$$y[n] = \alpha_1 y[n-1] + \alpha_2 y[n-2] + \dots + \alpha_M y[n-M]$$

General Solution

$$y[n] = \sum_{i=1}^{M} A_i \lambda_i^n \quad \text{where}$$

Natural Frequencies → λ_i's are the roots of

$$\lambda^M - \alpha_1 \lambda^{M-1} - \alpha_2 \lambda^{M-2} + \dots + \alpha_{M-1}\lambda^1 + \alpha_M = 0$$

A_i's are selected to match initial conditions

Now consider an input $x[n]$

$$y[n] = \sum_{i=1}^{M} \alpha_i y[n-i] + \sum_{j=0}^{M} \beta_j x[n-j]$$

Note

Switch Formats for easier Analysis

$$\sum_{i=0}^{M} a_i\, y[n+i] = \sum_{j=0}^{M} b_j\, x[n+j]$$

Note Conversion to Normal Form

$$y[n] = -\sum_{i=1}^{M} \frac{a_{M-i}}{a_M}\, y[n-i] + \sum_{j=0}^{M} \frac{b_{M-j}}{a_M}\, x[n-j]$$

Why?

Natural frequencies are roots of

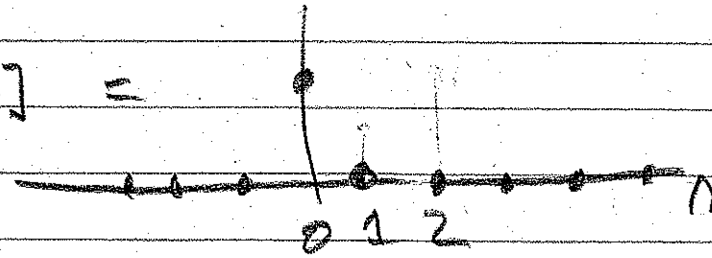
$$\sum_{i=0}^{M} a_i \lambda^i = 0$$

Simpler than

$$\lambda^M - \sum_{i=1}^{M} \alpha_i \lambda^{M-i} = 0$$

Are Natural Frequencies Important if there's an input

Suppose $x[n] =$



$$x[n] = 0 \quad n \neq 0$$
$$x[0] = 1$$

$$x[n] = \delta[n] \leftarrow \text{unit sample}$$

Example

$$y[n+1] - \frac{1}{2}y[n] = x[n] \qquad y[0] = 0$$

Natural Freq: $\lambda - \frac{1}{2} = 0 \quad \lambda = \frac{1}{2}$

$$y[n] = \frac{1}{2}y[n-1] - x[n-1] \leftarrow \text{Better for plug \& chug}$$

$$y[1] = \frac{1}{2}\overset{=0}{y[0]} + \overset{=1}{x[0]} = 1$$

$$y[2] = \frac{1}{2}\overset{=1}{y[1]} + \overset{=0}{x[1]} = \frac{1}{2}$$

$$y[3] = \frac{1}{2}y[2] = \left(\frac{1}{2}\right)^2$$

$$y[n] = \left(\frac{1}{2}\right)^n$$

Stability

Given $\sum_{k=0}^{m} a_k y[n+k] = \sum_{k=0}^{M} b_k x[n+k]$

If $y[n]$ in response to $x[n] = \delta[n]$ is such that

$$\lim_{n\to\infty} y[n] = 0$$

Then the system is stable

If roots of $\sum_{k=0}^{M} a_k \lambda^k = 0$ have magnitude less than 1

$\Rightarrow \lim_{n\to\infty} y[n] = 0$

Example

$$y[n+2] + 2y[n+1] + 2y[n] = x[n]$$

Natural Frequencies

$$\lambda^2 + 2\lambda + 2 = 0$$

$$(\lambda - (-1 + j))(\lambda - (-1 - j)) = 0$$

If $x[n] = \delta[n]$ then for large n

$$y[n+2] + 2y[n+1] + 2y[n] = 0$$

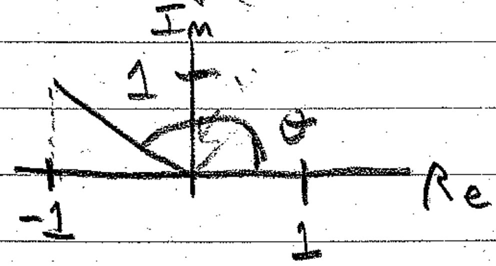
$$\Rightarrow \quad y[n] = A_1(-1 + j)^n + A_2(-1 - j)^n$$

for large n

change of representation $\Longrightarrow$ powers easier to evaluate

$$Me^{j\theta} = -1 + j \qquad (Me^{j\theta})^n = M^n e^{jn\theta}$$

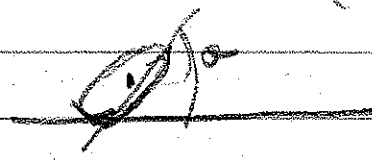
$$M = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$



$$\theta = \frac{3\pi}{4} \qquad \lim_{n \to \infty} y[n] \neq 0$$

(unstable system)

Composing Systems of Diff Eqns

Example Robot Steering 

Diff Eqn Model

$$d[n+1] = d[n] + \Delta t V \theta[n]$$

Approx

$$\theta[n+1] = \theta[n] + \Delta t \underbrace{X[n]}_{\text{input}}$$

Want to relate $d[n]$ to $X[n]$

Did it by shifting.

Suppose

$$\sum a_{w,k} W[n+k] = \sum b_{w,k} X[n+k]$$

$$\sum a_{v,k} V[n+k] = \sum b_{v,k} W[n+k]$$

$$\sum a_{y,k} Y[n+k] = \sum b_{y,k} V[n+k]$$

How will we generate

$$\sum a_k Y[n+k] = \sum b_k X[n+k] \text{ ?}$$

Need a change of representation!

Z - Transform (Bilateral)

$$\underbrace{\hat{X}(z)}_{\substack{\text{function}\\ \text{of } z}} = \sum_{n=-\infty}^{\infty} \underbrace{x[n]}_{\substack{\text{function}\\ \text{of}\\ n}} z^{-n}$$

Key Property

Suppose $y[n] = x[n+1]$

$$\hat{Y}(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n+1] z^{-n}$$

$$= z \underbrace{\left(\sum_{n=-\infty}^{\infty} x[n+1] z^{-(n+1)} \right)}_{= \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \hat{X}(z)}$$

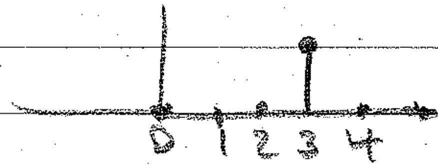
$$= z \hat{X}(z)$$

z transform of $x[n+k] = z^{k} \hat{X}(z)$

Useful Transforms

$x[n] = \delta[n]$ $\qquad$ $\hat{X}(z) = \sum \delta[n] z^{-n} = 1$

$x[n] = \delta[n-3]$



$\hat{X}(z) = \sum \delta[n-3] z^{-n} = z^{-3}$

Useful Transforms Continued

$$x[n] = \alpha^n \quad n \geq 0 \qquad \Rightarrow \tilde{X}(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$
$$x[n] = 0 \quad n < 0$$

$$= \sum (\alpha z^{-1})^n$$

$$= \frac{1}{1 - \alpha z^{-1}}$$

For a difference Eqn

$$\sum a_k y[n+k] = \sum b_k x[n+k]$$

$$\Rightarrow \left(\sum a_k z^k\right) \tilde{Y}(z) = \left(\sum b_k z^k\right) \tilde{X}(z)$$

$$\tilde{Y}(z) = \underbrace{\frac{\left(\sum b_k z^k\right)}{\left(\sum a_k z^k\right)}}_{\substack{\text{Transfer} \\ \text{function} \\ \text{(Rational function)}}} \tilde{X}(z)$$

Robot Steering with z

$$d[n+1] - d[n] = \Delta t \, V \, \theta[n]$$

$$\hat{D}(z) = \frac{\Delta t \, V}{z-1} \hat{\theta}(z)$$

$$\theta[n+1] - \theta[n] = \Delta t \, X[n]$$

$$\hat{\theta}(z) = \frac{\Delta t}{z-1} \hat{X}(z)$$

$$\hat{D}(z) = \frac{\Delta t \, V}{z-1} \frac{\Delta t}{z-1} \hat{X}(z)$$

$$= \frac{(\Delta t)^2 V}{z^2 - 2z + 1} \hat{X}(z)$$

Convert Back

$$(z^2 - 2z + 1) \hat{D}(z) = (\Delta t)^2 V \hat{X}(z)$$

$$d[n+2] - 2d[n+1] + d[n] = (\Delta t)^2 V \, x[n]$$

Steps

1) Convert to z
2) Manipulate transfer functions
3) Convert back to diff eqns

Block Diagram

$$\tilde{X}(z) \longrightarrow \boxed{\frac{\Delta t}{z-1}} \xrightarrow{\tilde{\Theta}(z)} \boxed{\frac{\Delta t V}{z-1}} \longrightarrow \tilde{D}(z)$$

$$\tilde{X}(z) \longrightarrow \boxed{\frac{(\Delta t)^2 V}{z^2-2z+1}} \longrightarrow \tilde{D}(z)$$

Now Suppose there's feedback

$$x[n] = K\Big(\underbrace{d_{desired}[n] - d[n]}\Big)$$

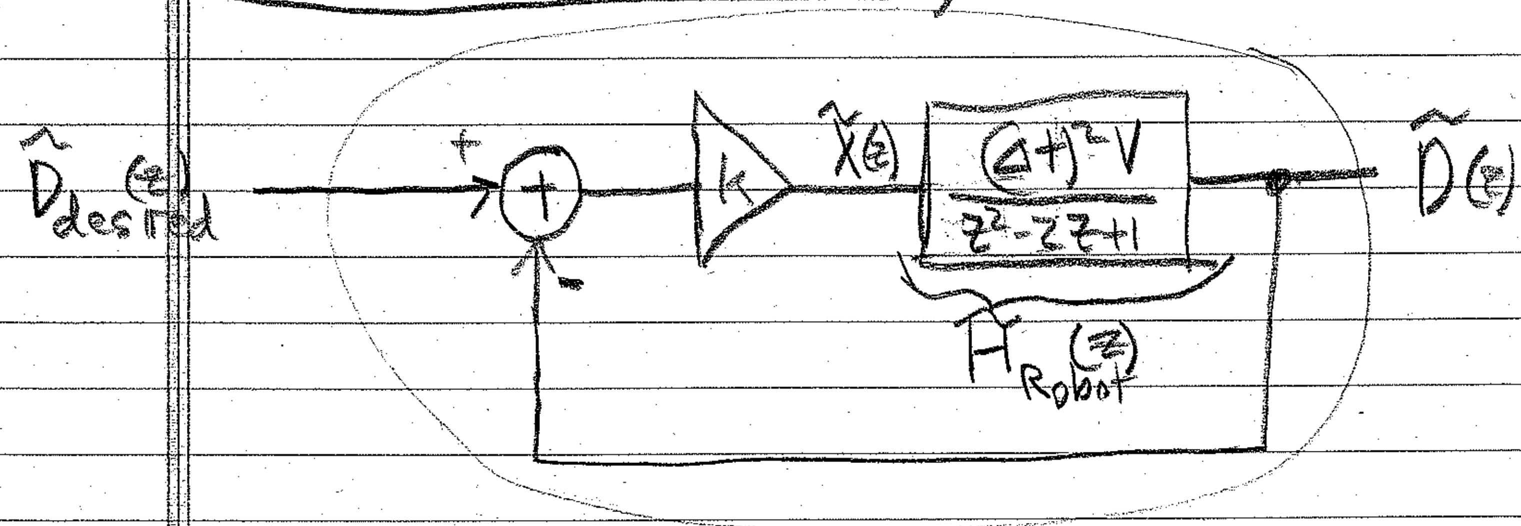
rotational velocity — $x[n]$

"gain" — K

difference between what you want and what you got

$$\Rightarrow \tilde{X}(z) = K\,\tilde{D}_{desired}(z) - \tilde{D}(z)$$

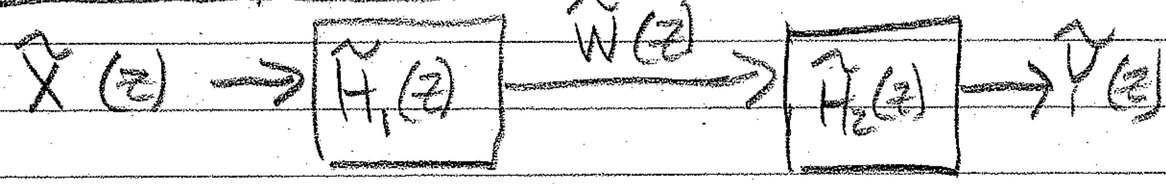
New Block Diagram



How do we get the transfer function from $\tilde{D}_{desired}$ to $\tilde{D}(z)$?

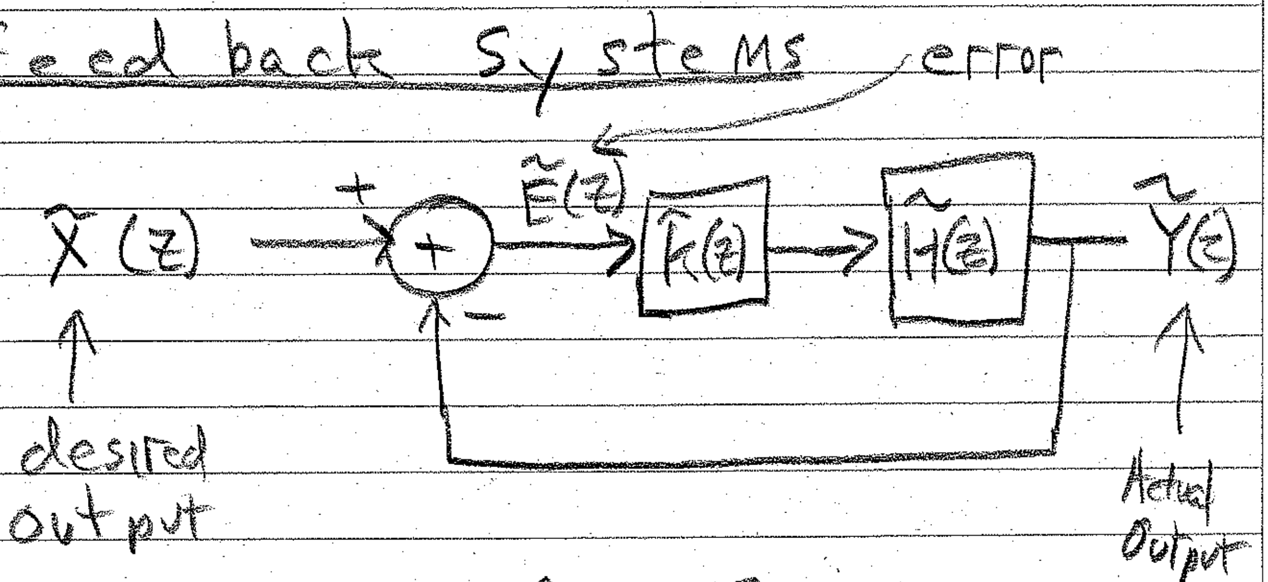
Two General Ideas

Cascaded Systems



$$\hat{Y}(z) = \hat{H}(z)\,\hat{X}(z) = \hat{H}_2(z)\,\hat{H}_1(z)$$

Feedback Systems



$$\hat{Y}(z) = \hat{H}(z)\,\hat{K}(z)\,\hat{E}(z)$$

$$\hat{E}(z) = \hat{X}(z) - \hat{Y}(z)$$

$$\hat{Y}(z) = \hat{H}(z)\,\hat{K}(z)\left(\hat{X}(z) - \hat{Y}(z)\right)$$

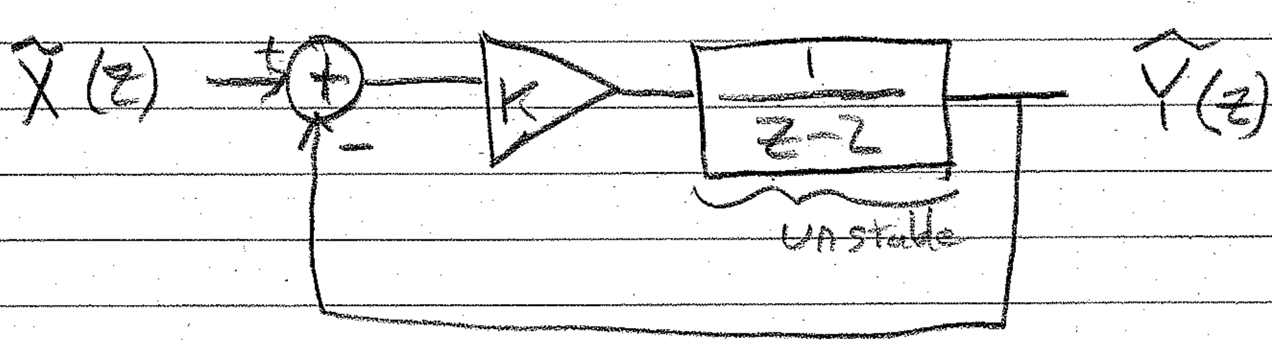
$$\hat{Y}(z) = \frac{\hat{H}(z)\,\hat{K}(z)}{1 + \hat{H}(z)\,\hat{K}(z)}\,\hat{X}(z)$$

Black's formula

Example

$$y[n+1] - 2y[n] = K\, e[n]$$

$$e[n] = x[n] - y[n]$$



$$\tilde{Y}(z) = \frac{K \frac{1}{z-2}}{1 + K\frac{1}{z-2}}\, \tilde{X}(z)$$

$$= \frac{K}{z-2+K}\, \tilde{X}(z)$$

$$\underbrace{y[n+1] + (K-2)\, y[n]}_{} = K\, x[n]$$

$|$natural freqs$| < 1$ if $|K-2| < 1$

Stabilize an unstable System

Black's Formula for Robot

$$\hat{D}(z) = \tilde{H}_{Robot}(z)\ \tilde{X}(z)$$

$$\tilde{X}(z) = K\,\hat{D}_{desired}(z) - \hat{D}(z)$$

$$\tilde{D}(z) = \tilde{H}_R(z)\ K\left(\tilde{D}_d(z) - \hat{D}(z)\right)$$

$$\hat{D}(z) = \frac{K\tilde{H}_R(z)}{1 + K\tilde{H}_R(z)}\ \tilde{D}_d(z)$$

$$\hat{D}(z) = \frac{K\,\frac{(\Delta t)^2 V}{z^2 - 2z + 1}}{1 + K\,\frac{(\Delta t)^2 V}{z^2 - 2z + 1}}\ \tilde{D}_d(z)$$

$$\tilde{D}(z) = \frac{K\Delta t^2 V}{z^2 - 2z + (1 + K\Delta t^2 V)}\ \hat{D}_d(z)$$

$$d[n+2] - 2d[n+1] + (1 + K(\Delta t)^2 V)\,d[n] = K(\Delta t)^2 V\,d_d[n]$$

nat freqs : $1 \pm \sqrt{1 - (1 + K(\Delta t)^2 V)}$

$= 1 \pm \sqrt{-K(\Delta t)^2 V}$