6.081 10/17/06

Difference Equations

1) Uses
2) Homogeneous Case
   a) Simple First Order Example
   b) Geleral n^{th} order approach
   c) Natural Frequencies
   d) Stability

What is a difference equation

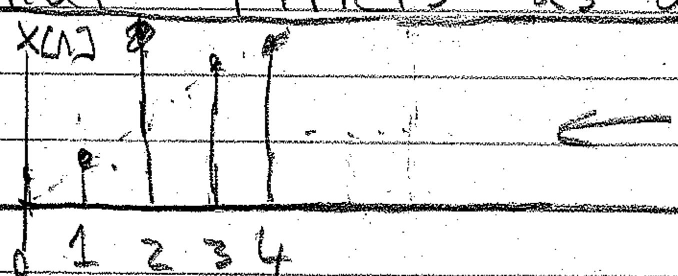
$$y[n] = f(y[n-1], y[n-2] \ldots, \underbrace{x[n], x[n-1] \ldots}_{\text{inputs}})$$

output — y

time index — $[n]$

Older Values — $y[n-1], y[n-2]$

Sonar Filters as an example



Sonar as robot moves away from wall

noise data (point 2)

Averaging $\left\{ \; y_A[n] = \frac{1}{2} x[n] + \frac{1}{2} x[n-1] \right.$

y_A — Filtered sonar

$x[n]$ — present reading

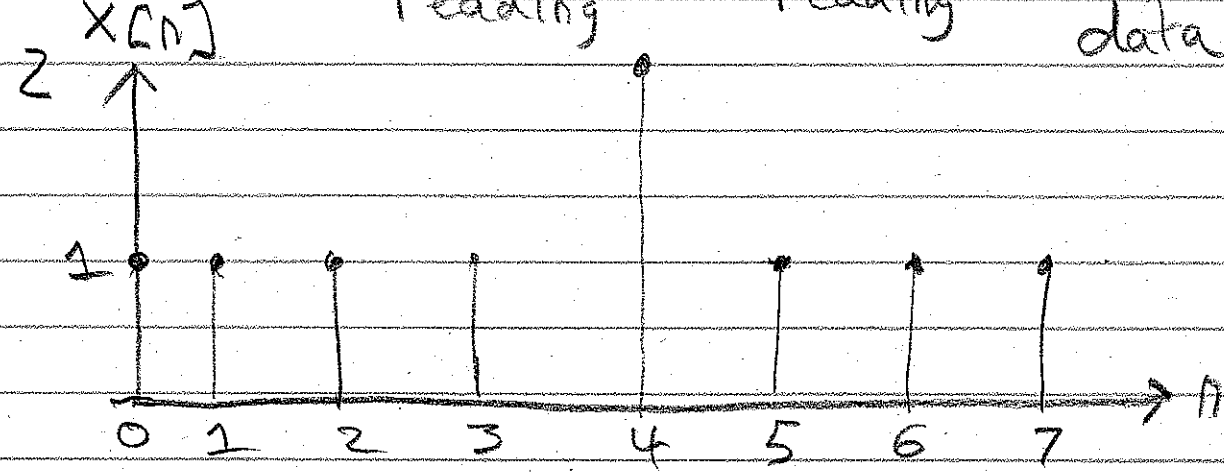
$x[n-1]$ — last reading

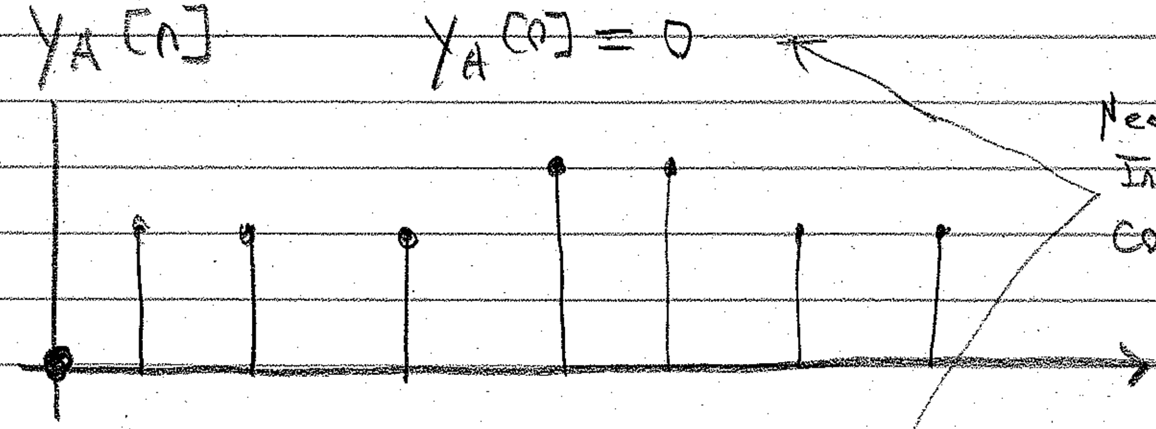
Smoothing $\left\{ y_s[n] = \frac{1}{2} y[n-1] + \frac{1}{2} x[n] \right.$

updated filtered reading

last filtered reading

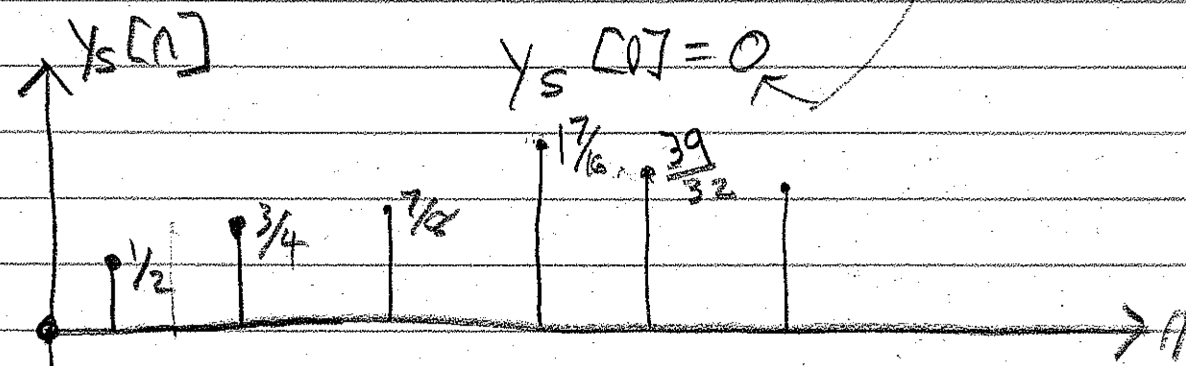
new data





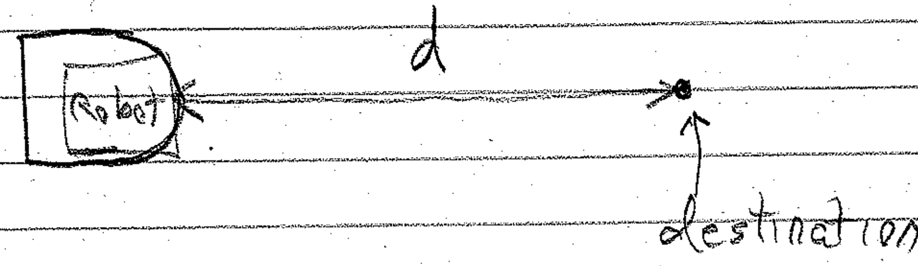
$y_A[0] = 0$

Needed Initial Conditions



$y_s[0] = 0$

Another Example   Homogeneous (No Input)



gain

$$d[n] = d[n-1] + \Delta t \, \underbrace{K \, d[n-1]}_{\text{velocity}}$$

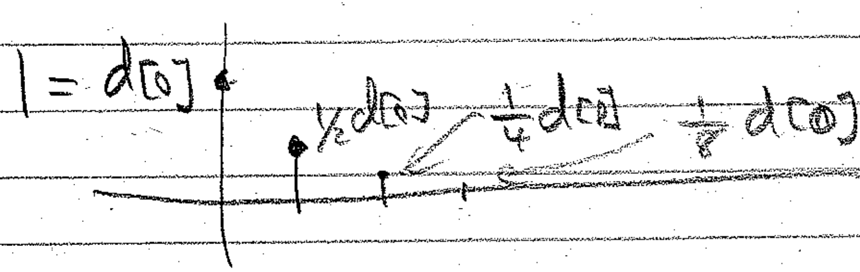
robot motor($K d[n-1]$, 0)

time between samples

$$d[n] = (1 + \Delta t K) \, d[n-1]$$

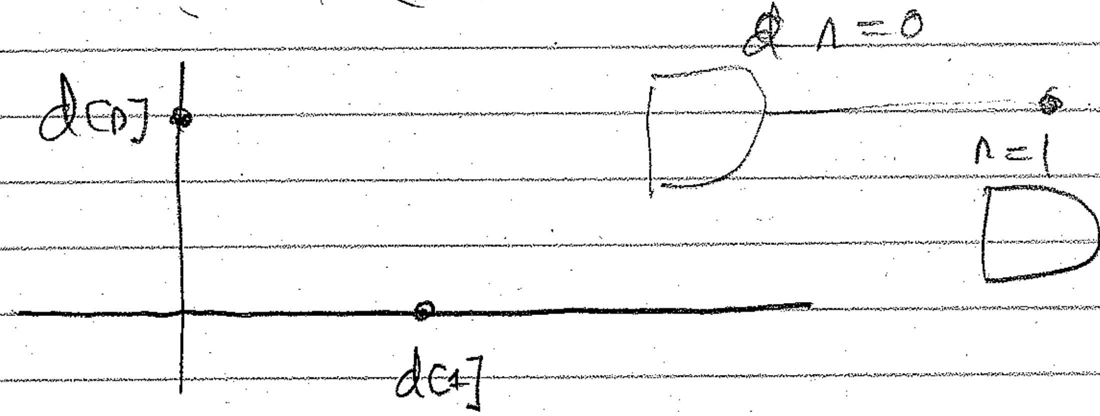
Case $(K \Delta t) = -0.5$

$$d[n] = +\frac{1}{2} d[n-1]$$



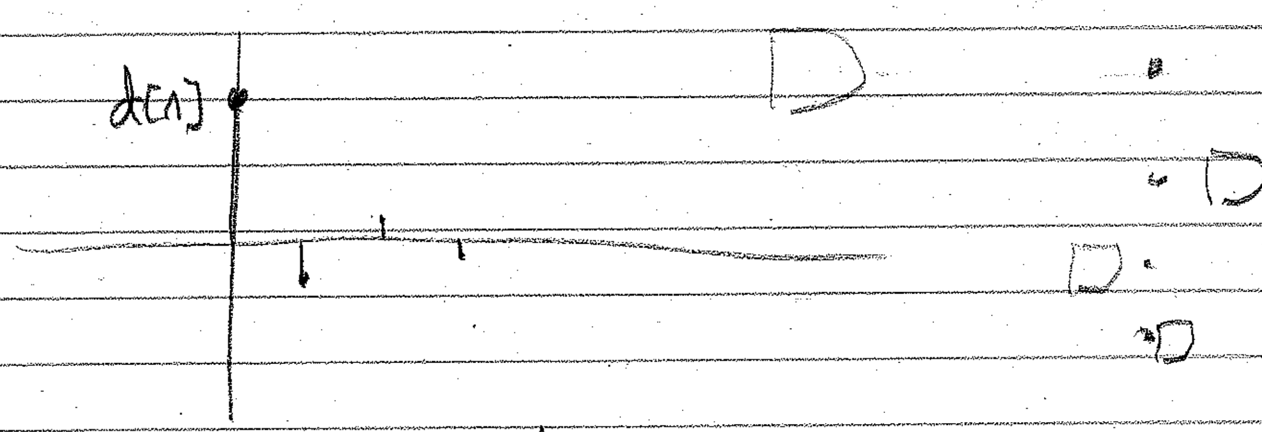
$(K\Delta t) = -1$

$d[n] = (1-1)\, d[n-1] = 0$

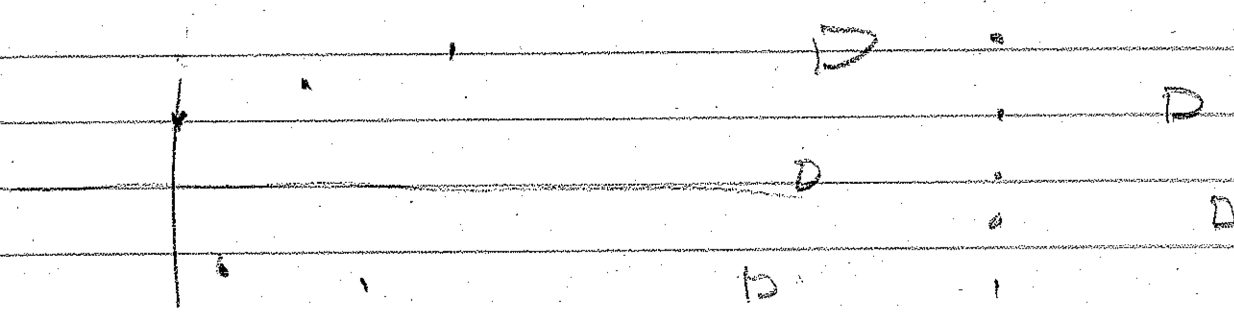


$(K\Delta t) = -1.2$

$d[n] = -0.2\, d[n-1]$



$K\Delta t = -2.1$



Another Homogenous Problem

Bank Interest:

Interest 1% compounded quarterly

$$X[n] = 1.01\, X[n-1]$$

$X[n]$: "new" quarter $X[n-1]$: last quarter

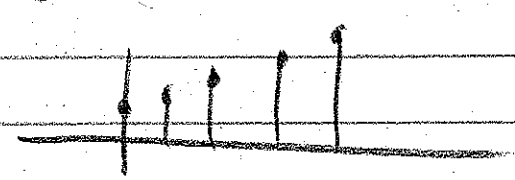
$$X[1] = 1.01\, X[0]$$
$$X[2] = (1.01)^2\, X[0]$$
$$X[3] = (1.01)^3\, X[0]$$

First Order Difference Eqns

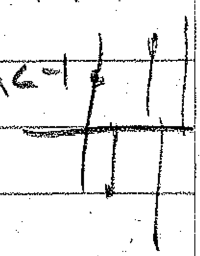
$$X[n] = \lambda X[n-1] \qquad X[0] = X_0$$

Soln $\quad X[n] = \lambda^n X[0] = \lambda^n X_0$

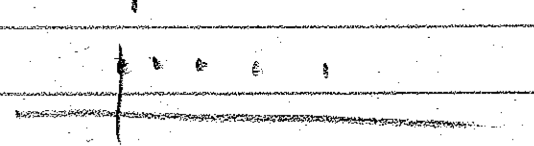
if $\lambda > 1$



if $\lambda < -1$

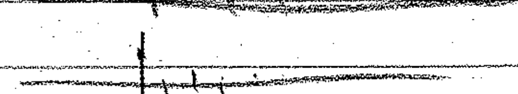


if $\lambda = 1$

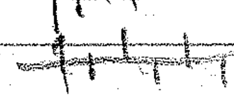


if $0 < \lambda < 1$

if $-1 < \lambda < 0$



if $\lambda = -1$



Why is λ important

1) Tells you qualitative info indep of initial condition

(soln grows, decays, oscillates, etc)

2) More than just computing response to a single initial condition!

3) Can Design for a good λ (Picking K for "stop at a point")

Robot Stop at a Point

$$d[n] = d[n-1] + \Delta t K d[n-1]$$

$$\lambda = 1 + \Delta t K$$

K should be negative

$$|K| < \frac{1}{\Delta t}$$

More Complicated Linear Diff Eqs

Fibonacci Numbers

$$\underbrace{x[n] = x[n-1] + x[n-2]}_{\text{Second order}}$$

Need two Initial conditions!

Stopping at a point

current velocity

$$d[n] = d[n-1] + \Delta t\,(K_1 d[n-1] + K_2 (d[n-1] - d[n-2]))$$

$$= (1 + \Delta t(K_1 + K_2))\, d[n-1] - \Delta t K_2\, d[n-2]$$

In General

$$x[n] = \underbrace{\alpha_1 x[n-1] + \alpha_2 x[n-2] + \dots + \alpha_m x[n-m]}_{\text{need } m \text{ initial conditions}}$$

What is the generalization of λ?

<u>Assume</u> $x[n] = A\lambda^n$

<u>Plug in</u>

$$\cancel{A}\lambda^n = \alpha_1 \cancel{A}\lambda^{n-1} + \alpha_2 \cancel{A}\lambda^{n-2} \ldots + \cancel{A}\lambda^{n-M}$$

<u>or</u>

$$p_M(\lambda) = \lambda^n - \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \ldots + \alpha_M \lambda^{n-M} = 0$$
$$= \lambda^M - \alpha_1 \lambda^{M-1} + \alpha_2 \lambda^{M-2} + \ldots + \alpha_M = 0$$

<u>So</u> if λ_i is a root of $p_M(\lambda)$

$$x[n] = A\lambda_i^n \quad \text{for any } A.$$

<u>Typically</u> There are M roots to $p_M(\lambda)$

$$x[n] = A_1 \lambda_1^n + A_2 \lambda_2^n + \ldots A_M \lambda_M^n$$

roots of $p_M(\lambda)$

How do we determine the A_i's?

Match Initial Conditions

Fib Example

$$x[n+2] - x[n+1] - x[n] = 0$$

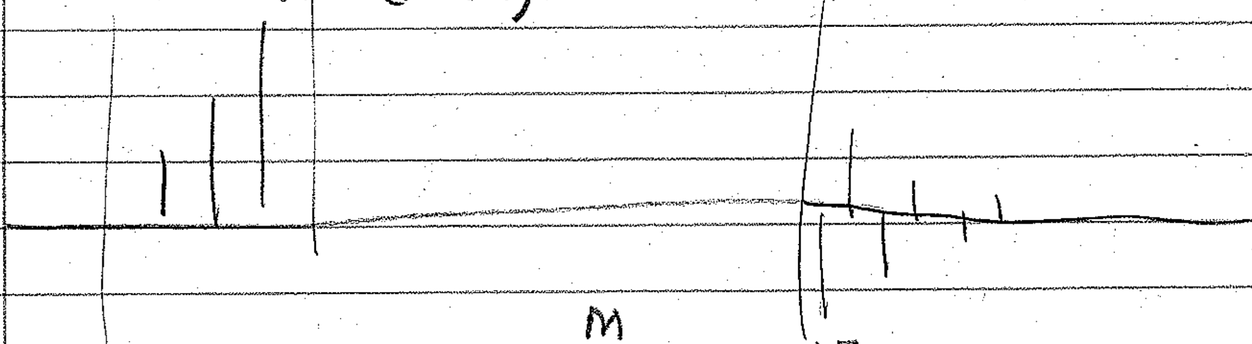
$\text{roots}(\lambda^2 - \lambda - 1) = 1.6180, \ -0.6180$

nat. freqs

$$A\,(1.618)^{n=0,\,n=1} + B\,(-0.6180)^{n=0,\,n=1} = 0, 1$$

$0.4472 \qquad -0.4472$

4181 $(n=19)$



Steps $\quad x[n] = \sum_{i=1}^{M} \alpha_i x[n-i]$

1) Determine roots of $\lambda^M + \sum_{i=1}^{M} \alpha_i \lambda^{M-i} = 0$

2) Solve for coefficients A_i by matching initial conditions

$$\sum A_i \lambda^0 = x[0]$$

$$\sum A_i \lambda^1 = x[1]$$

$$\sum A_i \lambda^{M-1} = x[M-1]$$

Another Example

$$X[n] = \sqrt{3}\, X[n-1] - X[n]$$

$$\lambda^2 - \sqrt{3}\lambda + 1 = \left(\lambda - \left(\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}j\right)\right)\left(\lambda - \left(\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}j\right)\right)$$

$$X[n] = A_1 \left(\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}j\right)^n + A_2 \left(\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}j\right)^n$$

Aside on Complex representation

$a + bj$ can be represented as

$$Me^{j\theta}$$

$\mathrm{Re}(Me^{j\theta}) = M\cos\theta = a$ $\mathrm{Im}(Me^{j\theta}) = M\sin\theta = b$

Key Reason

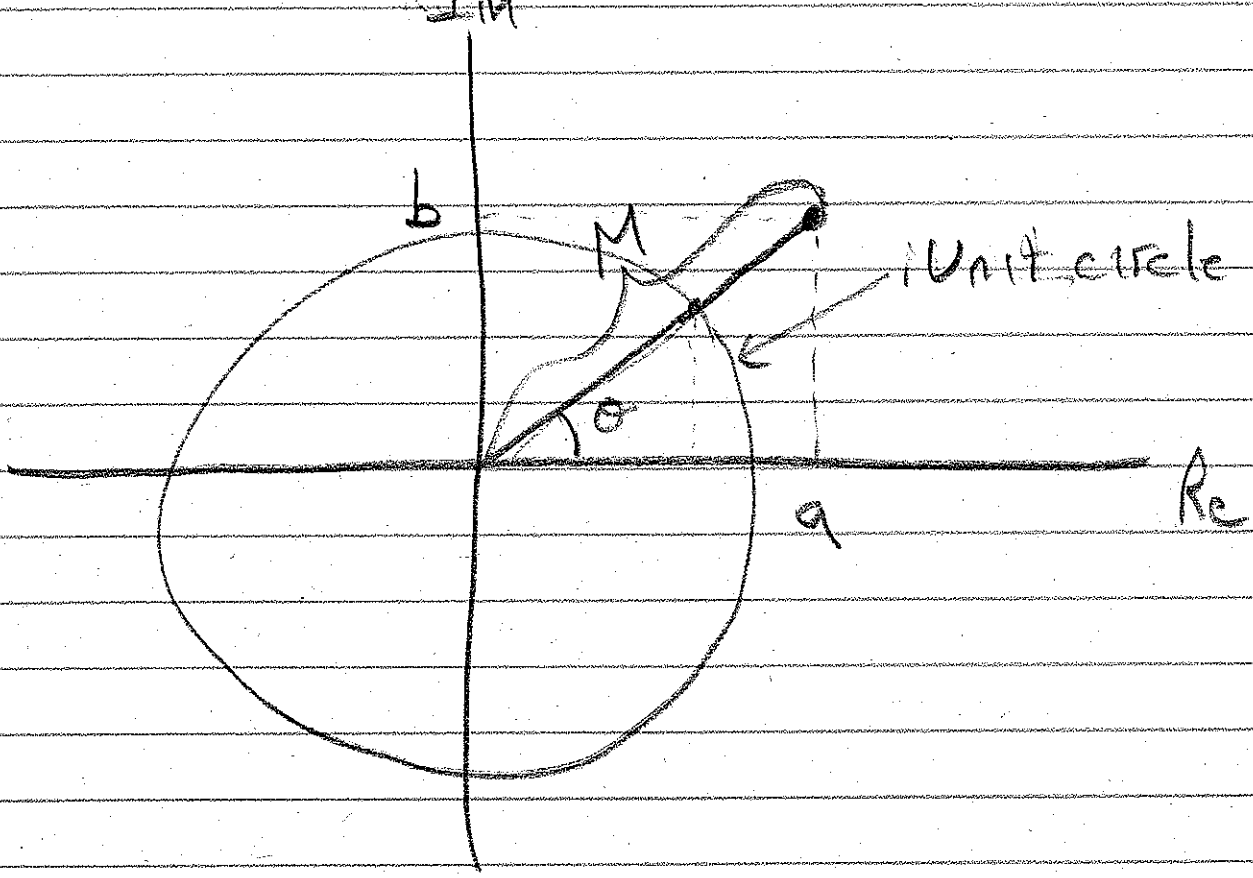
$$(Me^{j\theta})^2 = Me^{j\theta}\, Me^{j\theta} = M^2 e^{j2\theta}$$

$$(Me^{j\theta})^n = M^n e^{jn\theta}$$

Key Question

$$\lim_{n \to \infty} (Me^{j\theta})^n = \lim_{n \to \infty} M^n e^{jn\theta} = ?$$

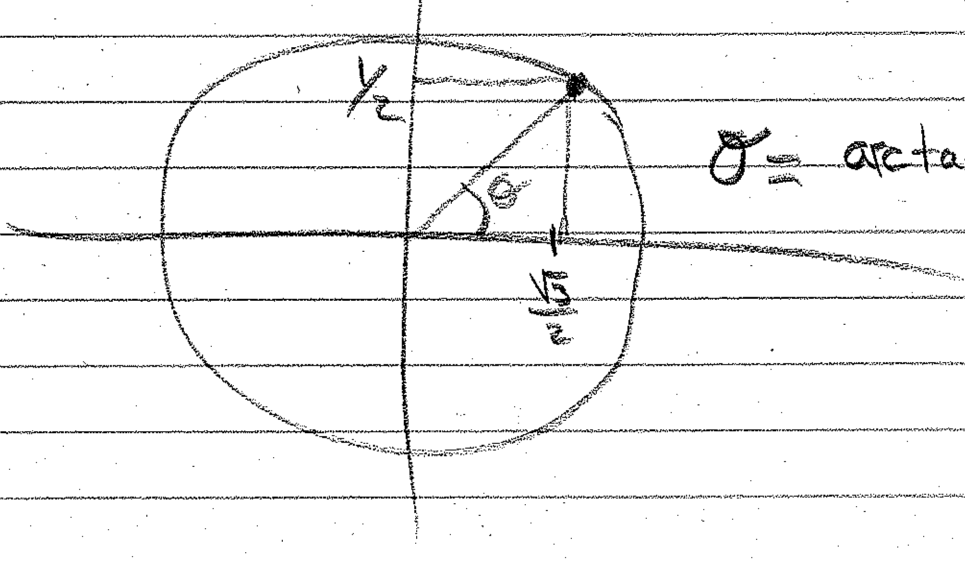
Interpretation



$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}j\right)$$

$$M = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$



$$\theta = \arctan\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \frac{\pi}{6}$$

For a M-th order Difference Eqn

1) $\lambda^M + \sum_{i=1}^{M} \alpha_i \lambda^{M-i} = 0$ has M roots (Usually)

2) $\lambda_1, \ldots \lambda_M$ are the natural frequencies

3) If $|\lambda_i| < 1$ decays

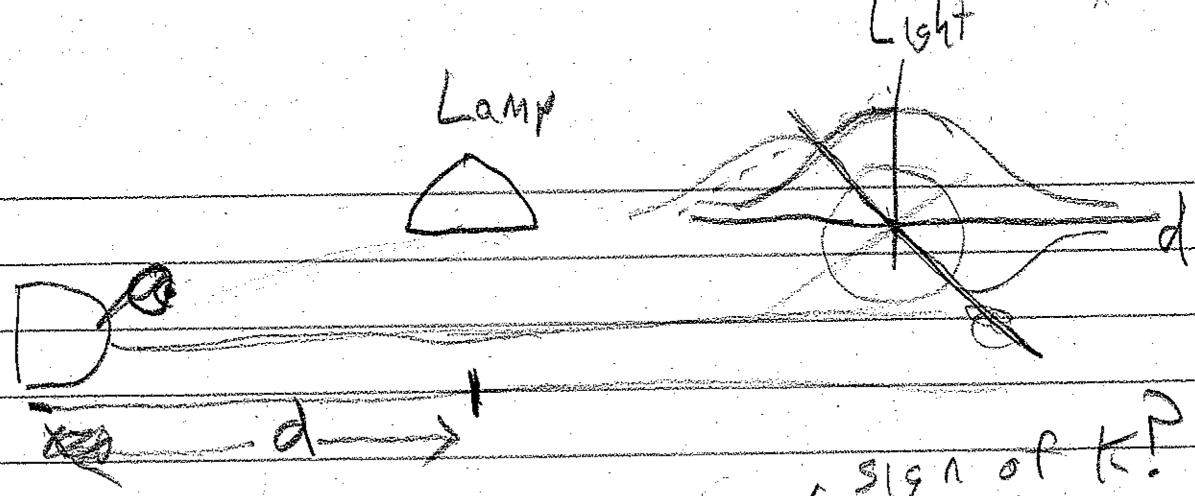
If $|\lambda_i| > 1$ grows

If $|\lambda_i| = 1$ oscillates forever

Stability

$$|\lambda_i| < 1$$

x^2



sign of K?

$$d[n] = d[n-1] + \Delta t K \left(L[n-1] - L[n-2] \right)$$

$$\approx \Delta t K A \left(d[n-1] - d[n-2] \right)$$

$$d[n] = (1 + \Delta t K A)\, d[n-1] - \Delta t K A\, d[n-2]$$

$$d[n] - (1 + \Delta t K A)\, d[n-1] + \Delta t K A\, d[n-2] = 0$$

$$\frac{-(1+\Delta t K A) \pm \sqrt{(1+\Delta t K A)^2 - 4(\Delta t K A)}}{2}$$