6.006 - Introduction to Algorithms

Lecture 17:

Heuristics for Faster Graph Search

Linear time is too slow...

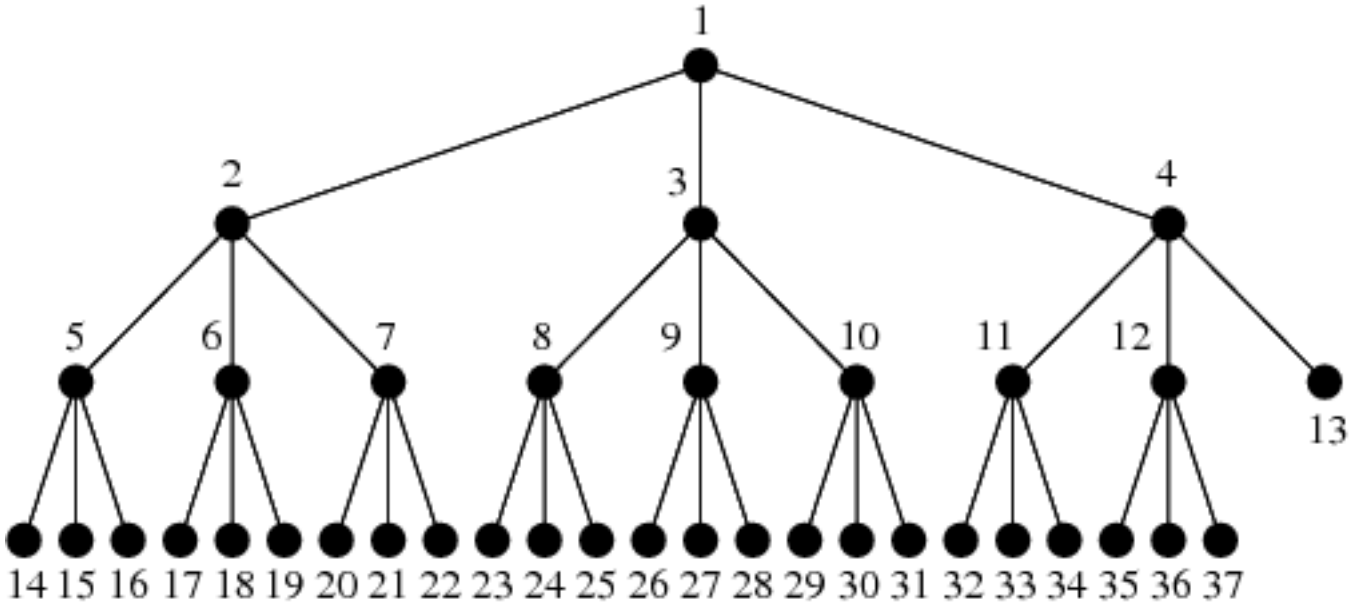
- Google Maps: ~10^{10} locations, 10^{11} edges
- Dijkstra's would take Θ(1 minute)

Today's goals

- Develop heuristics for shortest path searches
  - Preserve correctness
  - Improve runtime in practice, not in theory
- Consider special classes of graphs:
  - Random graphs
  - Planar-weighted graphs

Part 1: “random” graphs

- Every vertex has *d* random neighbors
- Consider the neighborhood of a vertex *s*
  - Number of vertices at distance 1: *d*
  - Number at distance 2: ~d^2
  - ...number at distance *k*: ~d^k



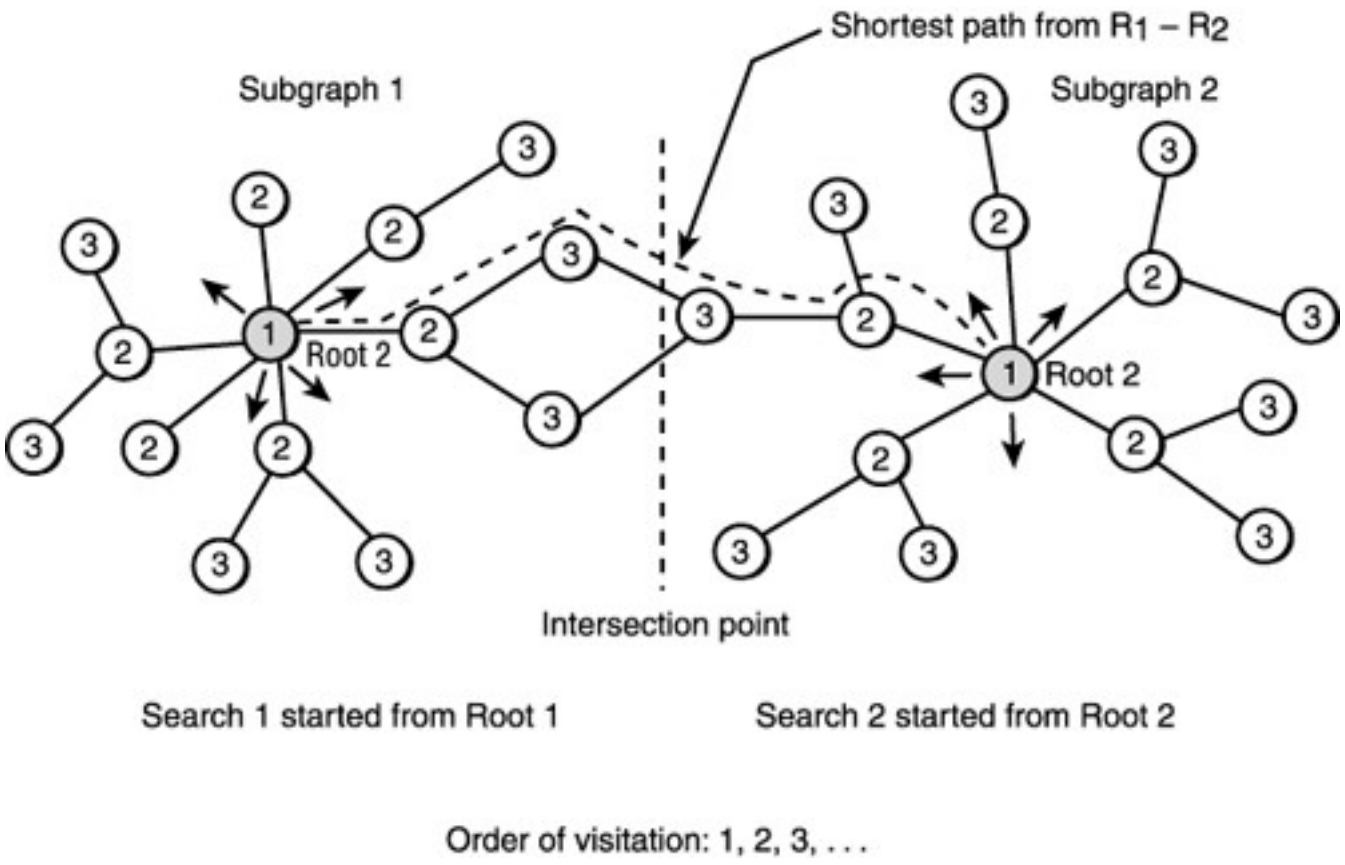
BFS in random graphs

- *G* is a random graph (*n* vertices, degree *d*)
- Suppose we search for a path from *s* to *t* in G
  - Almost all vertices are at levels ~$\log_d n$
  - Almost all time spent at the last levels
- How can we improve our runtime?

Bidirectional BFS

- Idea: instead of running a BFS from *s* to *t*, run BFS from *s* to *t* and from *t* to *s* simultaneously
  - For each level *i*:
    - Compute vertices at distance *i* from *s*
    - Compute vertices at distance *i* from *t*
  - Stop when a vertex *v* has been found from both *s* and *t*
  - Shortest path from *s* to *t* runs through *v*

Example of bidirectional BFS



Order of visitation: 1, 2, 3, . . .

Proof of correctness

- If shortest path from s to t is of length $2k$, then middle vertex v_k appears in both level ks
- If shortest path is of length $2k+1$, then vertex v_{k+1} appears in s-level $k+1$ and t-level k
- Is this too easy?

“Analysis” on random graphs

- Bidirectional BFS expands $(\log_d n) / 2$ levels, instead of $\log_d n$
  - Explores about $\sqrt{n}$ vertices
  - Graph search in sublinear time!
- Performs well on many non-random graphs

Bidirectional Dijkstra

- Run Dijkstra simultaneously forwards from *s* and backwards to *t*
- Keep vertices in two min-heaps:
  - First sorted by distance from *s*
  - Second sorted by distance to *t*
- Pop the smaller of the two minimums
  - From *s* heap: add it to a set *S*
  - From *t* heap: add it to *T*
- Repeat till we add a vertex *v* to both sets

Subtleties in bidirectional Dijkstra

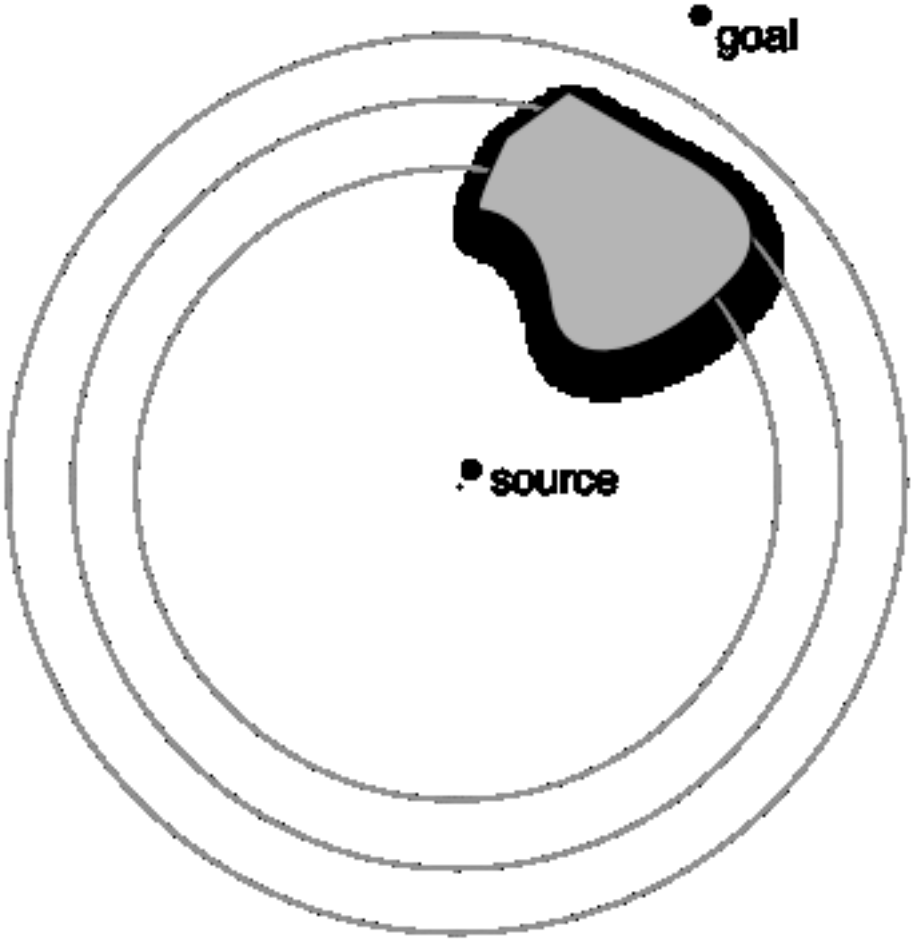
- The shortest path from *s* to *t* does not necessarily run through the vertex *v*...
  - It goes from something in *S* to something in *T*
- Loop over every edge from a vertex *x* in *S* to a vertex *y* in *T*
  - Find paths with lengths d(*s*, *x*) + *l*(*x*, *y*) + d(*y*, *t*)
  - If any of these paths is shorter than the path through *v* (d(*s*, *v*) + d(*v*, *t*)), return it instead

Part 2: planar-weighted graphs

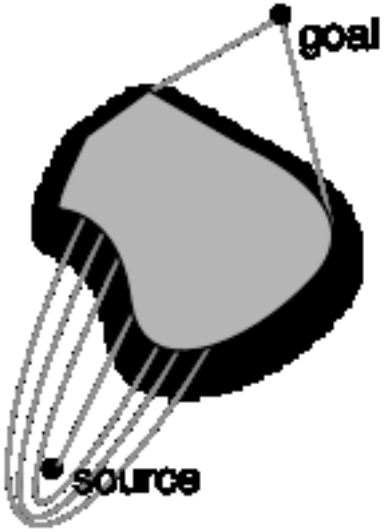
- In a planar-weighted graph, vertices are points
- Edge length *l*(*u*, *v*) is the distance from *u* to *v*
- We've seen this before:

Dijkstra on planar-weighted graphs

- In reality:



- In an ideal world:



Goal-directed search: A*

- Idea: use extra information to guide search from s to t
- Assign each vertex v a potential $\lambda(v)$
  - t should have potential $\lambda(t) = 0$
  - Vertices close to t should have low potential
- Try to search toward low potential
  - Modify edge costs: $l'(u, v) = l(u, v) - \lambda(u) + \lambda(v)$
  - Run Dijkstra?

Edge modification preserves paths

- New edge costs: $l'(u, v) = l(u, v) - \lambda(u) + \lambda(v)$
- Claim: the shortest path from u to v is preserved by edge modification
  - Let $(u, v_1, v_2, \ldots v_k, v)$ be a path from u to v
  - New path length:

$$l'(u, v_1) + l'(v_1, v_2) + \ldots + l'(v_k, v)$$

$$= l(u, v_1) - \lambda(u) + \lambda(v_1) + l(v_1, v_2) - \lambda(v_1) + \lambda(v_2) + \ldots + l(v_k, v) - \lambda(v_k) + \lambda(v)$$

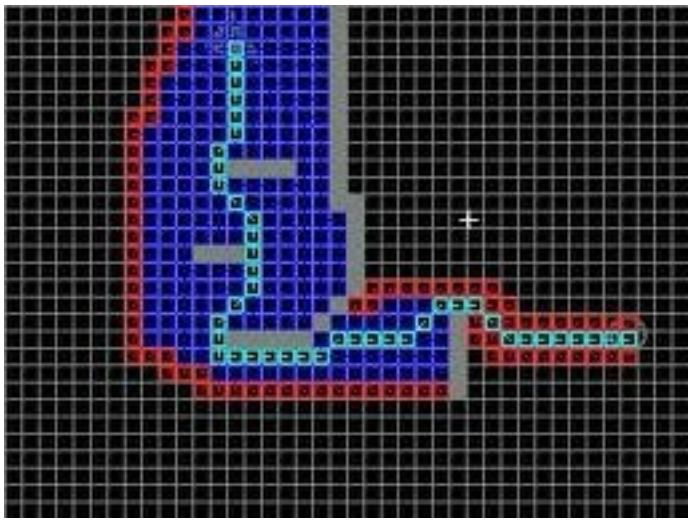
$$= [l(u, v_1) + l(v_1, v_2) + \ldots + l(v_k, v)] - \lambda(u) + \lambda(v)$$

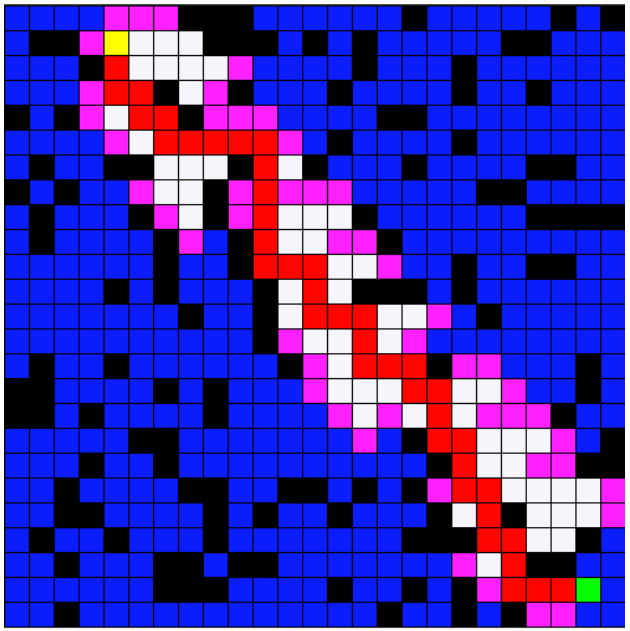
- New path length = old path length $- \lambda(u) + \lambda(v)$

Consistent heuristics

- Edge modification preserves paths
- We can use Dijkstra if $l'(u, v) \geq 0$ for all u, v
  - As long as $l(u, v) - \lambda(u) + \lambda(v) \geq 0$
- How to choose $\lambda(u)$?
  - Suppose graph is planar-weighted
  - Use distance to t as potential: $\lambda(u) = d(u, t)$
  - Triangle inequality: $l(u, v) + d(v, t) \geq d(u, t)$
- Other graphs – other potentials

Results of A*





A* has been called one of the top ten algorithms of the last century!

Other ideas to speed up search...

- Precompute shortest paths for some pairs...
- “Incremental”: use data from prior searches...
- Only return approximate shortest paths...
- …