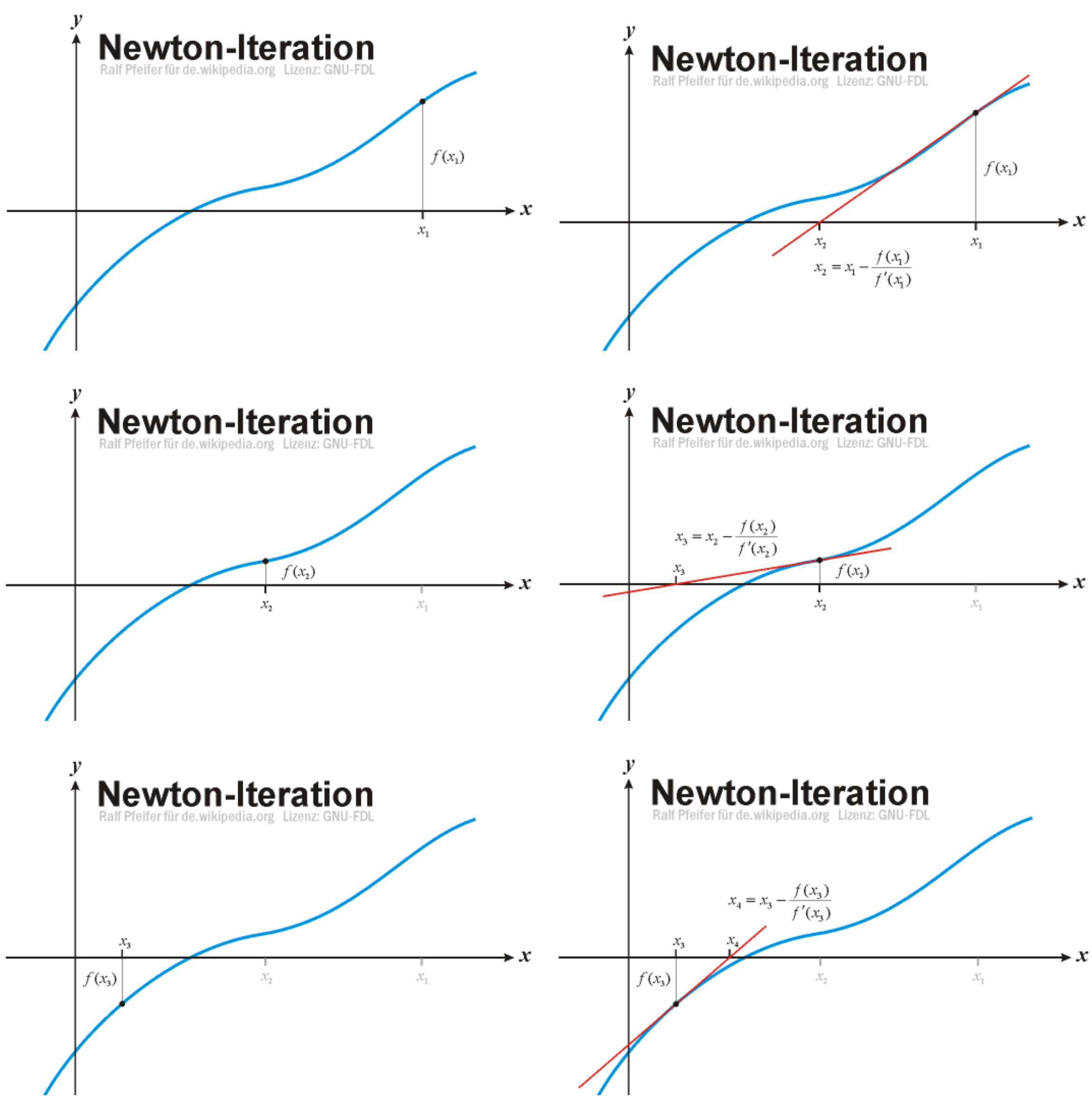
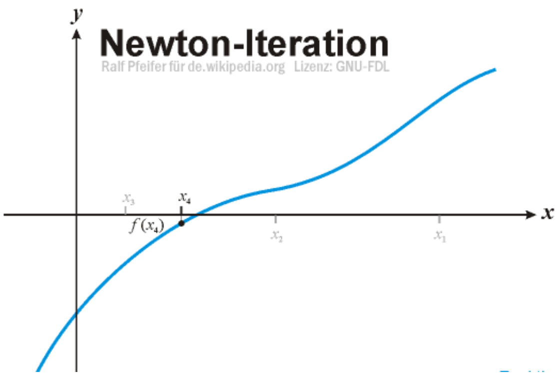
Newton's Method

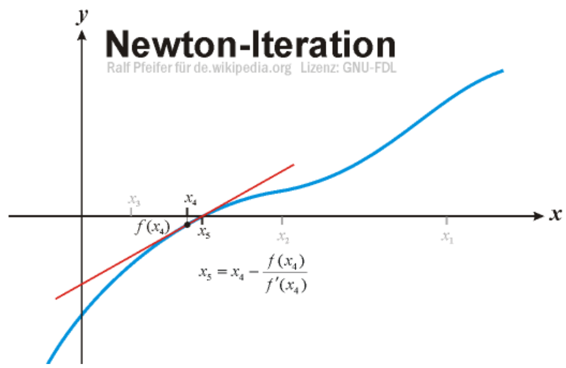
Newton's method is a method that iteratively computes progressively better approximations to the roots of a real-valued function $f(x)$. Its input is an initial guess x_0 and the function $f(x)$. The method goes as follows:

1. Given x_0 and $f(x)$, initialize $n = 0$
2. Compute $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
3. Repeat step 2 until $f(x_n)$ is sufficiently close to a root of $f(x)$.

An example of Newton's method in action:







Deriving Heron's Method

Heron's method (or the Babylonian method) is an algorithm that approximates $\sqrt{S}$. We can interpret this problem as solving for the roots of the function $f(x) = x^2 - S$. Since $\sqrt(S)$ is a zero for this problem, we can apply Newton's method to derive a method to solve for square roots.

In this particular case, $f(x_n) = x_n^2 - S$ and $f'(x_n) = 2x_n$. Plugging this into Newton's method, we get the iterative step

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} && (1)\\
&= x_n - \frac{x_n^2 - S}{2x_n} && (2)\\
&= x_n - \frac{x_n^2}{2x_n} + \frac{S}{2x_n} && (3)\\
&= x_n - \frac{x_n}{2} + \frac{S}{2x_n} && (4)\\
&= \frac{x_n}{2} + \frac{S}{2x_n} && (5)\\
&= \frac{1}{2}(x_n + \frac{S}{x_n}) && (6)
\end{aligned}
$$

If we compute $y_n = \frac{S}{x_n}$ on the side for each step, we can reword our method. Given initial guess $x_0 = 1$:

$$
\begin{aligned}
y_n &= \frac{S}{x_n} && (7)\\
x_n + 1 &= \frac{x_n + y_n}{2} && (8)
\end{aligned}
$$

As we iterate, both x_n and y_n will converge to $\sqrt{S}$.

Division and Newton's Method

We've seen how to multiply two numbers in previous lectures and recitations, but division is a little bit more tricky. Given two numbers N and D, we want to find the quotient N/D. Since we know how to multiply, our goal is to find a method to calculate $1/D$ and then compute the product of $1/D$ and N to get our desired result. To calculate $1/D$, we can use Newton's method.

The input function would be $f(x) = \frac{1}{x} - D$. This sets our iterative step to be

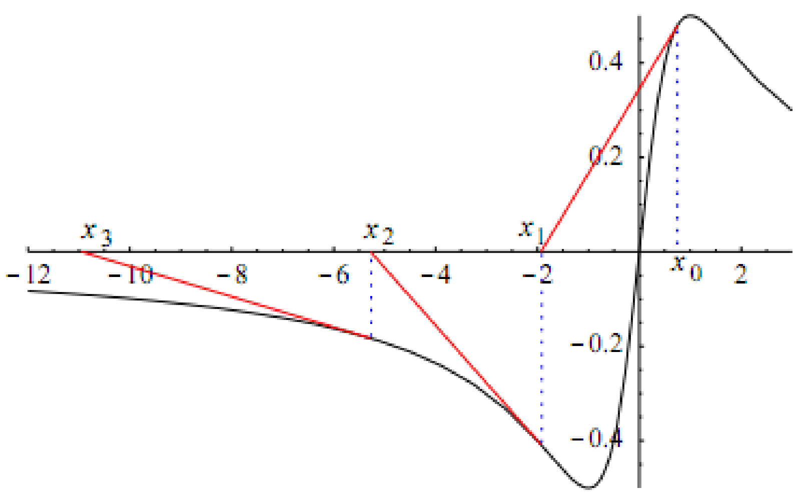
$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} && (9) \\ &= x_n - \frac{(1/x_n) - D}{-1/x_n^2} && (10) \\ &= x_n + x_n(1 - Dx_n) && (11) \\ &= x_n(2 - Dx_n) && (12) \end{aligned}$$

Once we've iterated enough, we will get an approximation of $1/D$. Using our multiplication methods from before, we can use this result to solve N/D.

Limitations to Newton's Method

Sometimes calculating the derivative of $f(x)$ is not easily done analytically if $f(x)$ is a complex function. In this case, it may be difficult to compute $f'(x_n)$ for each step of Newton's method. However, we can approximate $f'(x_n)$ by calculating $f(x_n + \epsilon)$ and $f(x_n - \epsilon)$ and finding the slope between those two points surrounding $f(x_n)$.

In some cases, using Newton's method does not converge to a solution. In the example below, $f(x) = 0$ as x approaches $-\infty$. Though there's a root at $f(0)$, using Newton's method will miss this root in this case. Instead, the tangents follow the curve to the "root" at $f(-\infty)$. In this case, Newton's method diverges and will never converge to a root.



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