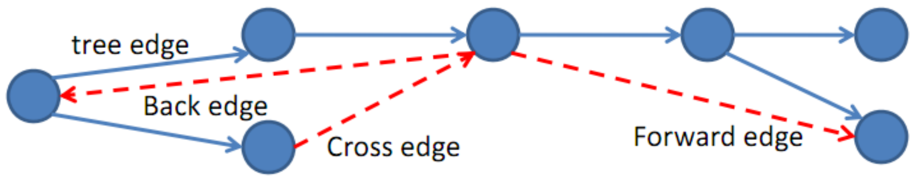
DFS Edge Classification

The edges we traverse as we execute a depth-first search can be classified into four edge types. During a DFS execution, the classification of edge (u, v), the edge from vertex u to vertex v, depends on whether we have visited v before in the DFS and if so, the relationship between u and v.

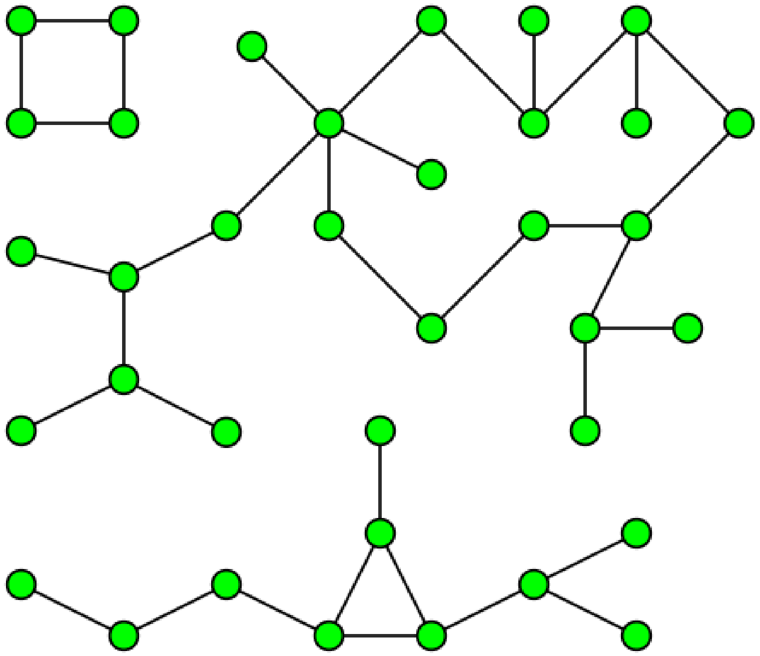
1. If v is visited for the first time as we traverse the edge (u, v), then the edge is a **tree edge**.
2. Else, v has already been visited:
   (a) If v is an ancestor of u, then edge (u, v) is a **back edge**.
   (b) Else, if v is a descendant of u, then edge (u, v) is a **forward edge**.
   (c) Else, if v is neither an ancestor or descendant of u, then edge (u, v) is a **cross edge**.



After executing DFS on graph G, every edge in G can be classified as one of these four edge types. We can use edge type information to learn some things about G. For example, **tree edges** form trees containing each vertex DFS visited in G. Also, G has a cycle if and only if DFS finds at least one **back edge**. Note that undirected graphs cannot contain **forward edges** and **cross edges**, since in those cases, the edge (v, u) would have already been traversed during DFS before we reach u and try to visit v.

Connected Components

A connected component is defined as a subgraph where there exists a path between any two vertices in it. Graph G is made up of separate connected components and it may be useful to be able to classify each vertex by which connected component it belongs to.



For undirected graph G, executing a BFS or DFS starting from a vertex v will visit every other vertex in the same connected component as v. We can mark every vertex visited from a BFS/DFS from v as being "owned" by v. As we iterate through all the vertices, we execute a BFS/DFS starting from a vertex if it has no owner (i.e. it is part of an undiscovered connected component) and mark all the vertices visited in that BFS/DFS. After iterating through all the vertices, each vertex will be marked by its owner, representing which connected component it is a part of. In summary, the algorithm is the following:

1. For each vertex v in undirected graph G
    (a) If v has no owner, it is part of an undiscovered connected component. Execute BFS or DFS starting from v and mark all the vertices as being owned by v
    (b) Else, if v has an owner, it is part of a connected component we've already discovered. Ignore v and move on to the next vertex.

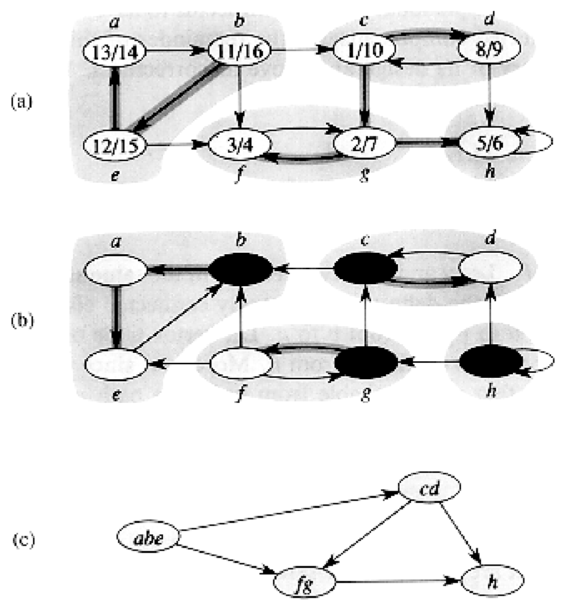
The runtime of this algorithm is $O(|V| + |E|)$ since each vertex is visited twice (once by iterating through it in the outer loop, another by visiting it in BFS/DFS) and each edge is visited once (in BFS/DFS).

Strongly Connected Components

The algorithm above does not work with directed graphs. For undirected graphs, finding a path from u to v implies that there exists a path from v to u. This is not the case for directed graphs. We can still separate the directed graphs into **strongly connected components**, which are components in directed graphs where any two vertices has a path in between each other. Note that this is the same definition as connected components above, but applied to directed graphs.

The intuition that will help us separate a directed graph into strongly connected components is realizing that a strongly connected component with its edges' directions reversed is still a strongly connected component. We will introduce G^T, which is the transpose of directed graph G. G^T and G are the same graph except the edge directions are reersed in G^T, i.e. if edge (u, v) is in G, then the edge (v, u) is in G^T. An algorithm to find strongly connected components goes as follows:

1. Execute DFS on G (starting at an arbitrary starting vertex), keeping track of the finishing times of all vertices
2. Compute the transpose, G^T
3. Execute DFS on G^T, starting at the vertex with the latest finishing time, forming a tree rooted at that vertex. Once a tree is completed, move on to the unvisited vertex with the next latest finishing time and form another tree using DFS and repeat until all the vertices in G^T are visited
4. Output the vertices in each tree formed by the second DFS as a separate strongly connected component



We can reduce a directed graph G to a graph of its strongly connected components, as seen above. Note that the graph of G's strongly connected components cannot contain a cycle, since a cycle of strongly connected components can itself be reduced into a single strongly connected component. We call a directed graph with no cycles a **dag**, short for directed acyclic graph. We can thus say that every directed graph G can be reduced to a dag of its strongly connected components.