Karatsuba’s Algorithm

6.006 Review Session
Problem Statement

• Given two n-digit long integers a and b in base r, find \( a \times b \).

• We’ve always assumed this is a constant time operation.
  – Makes life simpler.
  – Numbers are usually relatively small.
  – As we’ll see, we can do multiplication relatively fast.
Naïve Algorithm

• Using the algorithm we all love and know (the one we were taught in grade school) will take $O(n^2)$

• Would like to improve on this...
Divide and Conquer

• Let’s try divide and conquer.
  – Divide each number into two halves.
    • $x = x_H r^{n/2} + x_L$
    • $y = y_H r^{n/2} + y_L$
  – Then:
    \[
    xy = (x_H r^{n/2} + x_L) y_H r^{n/2} + y_L
    = x_H y_H r^n + (x_H y_L + x_L y_H) r^{n/2} + x_L y_L
    \]
  – Runtime?
    • $T(n) = 4 T(n/2) + O(n)$
    • $T(n) = O(n^2)$
Karatsuba’s Insight

• Instead of 4 subproblems, we only need 3 (with the help of clever insight).

• Three subproblems:
  - a = x_H y_H
  - d = x_L y_L
  - e = (x_H + x_L) (y_H + y_L) − a − d

• Then \( xy = a r^n + e r^{n/2} + d \)

• \( T(n) = 3 \ T(n/2) + O(n) \)

• \( T(n) = O(n^{\log_3 3}) = O(n^{1.584...}) \)
Worked Example

• Compute 1234 * 4321.

• Subproblems:
  – $a_1 = 12 \times 43$
  – $d_1 = 34 \times 21$
  – $e_1 = (12 + 34) \times (43 + 21) - a_1 - d_1$
    = $46 \times 64 - a_1 - d_1$
  – Need to recurse...
Worked Example

• First subproblem:
  \[ a_1 = 12 \times 43 \]

• Subproblems:
  \[ a_2 = 1 \times 4 = 4 \]
  \[ d_2 = 2 \times 3 = 6 \]
  \[ e_2 = (1+2)(4+3) - a_2 - d_2 = 11 \]

• Answer: \[ 4 \times 10^2 + 11 \times 10 + 6 = 516 \]
Worked Example

• Second subproblem
  \[ d_1 = 34 \times 21 \]

• Subproblems:
  – \[ a_2 = 3 \times 2 = 6 \]
  – \[ d_2 = 4 \times 1 = 4 \]
  – \[ e_2 = (3+4)(2+1) - a_2 - d_2 \]
    \[ = 11 \]

• Answer: \[ 6 \times 10^2 + 11 \times 10 + 4 = 714 \]
Worked Example

• Third subproblem:
  \[ e_1 = 46 \times 64 - a_1 - d_1 \]

• Subproblems:
  \[ a_2 = 4 \times 6 = 24 \]
  \[ d_2 = 6 \times 4 = 24 \]
  \[ e_2 = (4+6)(6+4) - a_2 - d_2 = 52 \]

• Answer: \[ 24 \times 10^2 + 52 \times 10 + 24 - 714 - 516 = 1714 \]
Worked Example

• Final Answer:

\[1234 \times 4321 = 516 \times 10^4 + 1714 \times 10^2 + 714\]

\[= 5,332,114\]