Quiz 2 Practice Problems

1 True/False

Decide whether these statements are **True** or **False**. You must briefly justify all your answers to receive full credit.

1. There exists a comparison sort of 5 numbers that uses at most 6 comparisons in the worst case.
   True **False**

   Explain:

2. Heapsort can be used as the auxiliary sorting routine in radix sort, because it operates in place.
   True **False**

   Explain:

3. If the DFS finishing time $f[u] > f[v]$ for two vertices u and v in a directed graph G, and u and v are in the same DFS tree in the DFS forest, then u is an ancestor of v in the depth first tree.
   True **False**

   Explain:

4. Let P be a shortest path from some vertex s to some other vertex t in a graph. If the weight of each edge in the graph is increased by one, P will still be a shortest path from s to t.
   True **False**

   Explain:

5. If an in-place sorting algorithm is given a sorted array, it will always output an unchanged array.
   True **False**

   Explain:

6. **[5 points]** Dijkstra's algorithm works on any graph without negative weight cycles.
   True **False**

   Explain:

7. **[5 points]** The Relax function never increases any shortest path estimate $d[v]$.
   True **False**

   Explain:

2 Short Answer

1. What property of the Rubik's cube graph made 2-way BFS more efficient than ordinary BFS?

2. What is the running time of the most efficient deterministic algorithm you know for finding the shortest path between two vertices in a directed graph, where the weights of all edges are equal? (Include the name of the algorithm.)

3 Topological Sort

Another way of performing topological sorting on a directed acyclic graph $G = (V, E)$ is to repeatedly find a vertex of in-degree 0 (no incoming edges), output it, and remove it and all of its outgoing edges from the graph. Explain how to implement this idea so that it runs in time $O(V + E)$. What happens to this algorithm if G has cycles?

4 Shortest Paths

Carrie Careful has hired Lazy Lazarus to help her compute single-source shortest paths on a large graph. Lazy writes a subroutine that, given $G = (V, E)$, a source vertex s, and a non-negative edge-weight function $w : E \to R$, outputs a mapping $d : V \to R$ such that $d[v]$ is supposed to be the weight $\delta(s, v)$ of the shortest-weight path from s to v (or ∞ if no such $s \to v$ path exists) and also a function $\pi : V \to (V \cup \{NIL\})$ such that $\pi[v]$ is the penultimate vertex on one such shortest path (or NIL if $v = s$ or v is unreachable from s).
Carrie doesn't trust Lazarus very much, and wants to write a "checker" routine that checks the output of Lazarus's code (in some way that is more efficient than just recomputing the answer herself).
Carrie writes a "checker" routine that checks the following conditions. (No need for her to check that $w(u, v)$ is always non-negative, since she creates this herself to pass to Lazarus.)

(i) $d[s] = 0$

(ii) $\pi[s] = NIL$

(iii) for all edges $(u, v) : d[v] \leq d[u] + w(u, v)$

(iv) for all vertices v : if $\pi[v] \neq NIL$, then $d[v] = d[\pi[v]] + w(\pi[v], v)$

(v) for all vertices $v \neq s$: if $d[v] < \infty$, then $\pi[v] \neq NIL$ (equivalently: $\pi[v] = NIL \implies d[v] = \infty$)

1. Show, by means of an example, that Carrie's conditions are not sufficient. That is, Lazarus's code could output some d, π values that satisfy Carrie's checker but for which $d[v] \neq \delta(s, v)$ for some v. (Hint: cyclic π values; unreachable vertices.)

2. How would you augment Carrie's checker to fix the problem you identified in (a)?

3. You are given a connected weighted undirected graph $G = (V, E, w)$ with no negative weight cycles. The *diameter* of the graph is defined to be the maximum-weight shortest path in the graph, i.e. for every pair of nodes (u, v) there is some shortest path weight $\delta(u, v)$, and the diameter is defined to be $\max\limits_{(u,v)}\{\delta(u, v)\}$.

   Give a polynomial-time algorithm to find the diameter of G. What is its running time? (Your algorithm only needs to have a running time polynomial in $|E|$ and $|V|$ to receive full credit; don't worry about optimizing your algorithm.)

4. You are given a weighted directed graph $G = (V, E, w)$ and the shortest path distances $\delta(s, u)$ from a source vertex s to every other vertex in G. However, you are not given $\pi(u)$ (the predecessor pointers). With this information, give an algorithm to find a shortest path from s to a given vertex t in $O(V + E)$ time.