6.006
Spring 2008
L24

Numerics II

Review: high precision arithmetic
Multiplication

Division
Algorithm
Error Analysis

Termination

Review

Want millionth digit of $\sqrt{2}$

$\lfloor \sqrt{2 \cdot 10^{2d}} \rfloor \qquad d = 10^6$

Compute $\lfloor \sqrt{a} \rfloor$ via Newton's method

$x_0 = 1$ (initial guess)

$$x_{i+1} = \frac{x_i + a/x_i}{2}$$

division!

Converges quadratically; # correct digits doubles each step.

MULTIPLICATION

1. Naive Divide & Conquer method: $\Theta(d^2)$ time
2. Karatsuba : $\Theta(d^{\log_2 3}) = \Theta(d^{1.584\ldots})$
3. Toom-Cook generalizes Karatsuba (break into $k \geq 2$ parts)
$$T(d) = 5T(d/3) + \Theta(d)$$
$$= \Theta(d^{\log_3 5}) = \Theta(d^{1.465\ldots})$$
4. Schonhage-Strassen - almost linear!
$\Theta(d \lg d \lg\lg d)$ using FFT.

All of these are in gmpy package

5. Furer (2007) : $\Theta(n \log n \, 2^{O(\log^* n)})$ where $\log^* n$ is iterated logarithm. # times log needs to be applied to get a number that is less than or equal to 1.

HIGH-PRECISION DIVISION

We want high-precision rep of $\frac{a}{b}$

- Compute high-precision rep of $\frac{1}{b}$ first
- High-precision rep of $\frac{1}{b}$ means $\left\lfloor \frac{R}{b} \right\rfloor$
  where R is large value s.t. it is easy to divide by R.

Ex: $R = 2^k$ for binary representations

Division

Newton's method for computing $\frac{R}{b}$

$$f(x) = \frac{1}{x} - \frac{b}{R} \qquad \left(\text{zero at } x = \frac{R}{b}\right)$$

$$f'(x) = \frac{-1}{x^2}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(\frac{1}{x_i} - \frac{b}{R}\right)}{-1/x_i^2}$$

$$x_{i+1} = x_i + x_i^2\left(\frac{1}{x_i} - \frac{b}{R}\right) = 2x_i - \frac{b x_i^2}{R}$$

easy div

multiply

EXAMPLE

Want $\frac{R}{b} = \frac{2^{16}}{5} = \frac{65536}{5} = 13107.2$

Try initial guess $\frac{2^{16}}{4} = 2^{14}$

$x_0 = 2^{14} = 16384$

$x_1 = 2 \cdot (16384) - 5(16384)^2 / 65536 = \underline{12288}$

$x_2 = 2 \cdot (12288) - 5(12288)^2 / 65536 = \underline{13056}$

$x_3 = 2 \cdot (13056) - 5(13056)^2 / 65536 = \underline{13107}$

ERROR ANALYSIS

Assume $x_i = \frac{R}{b}(1+\varepsilon_i)$

$$x_{i+1} = 2x_i - \frac{b x_i^2}{R}$$

$$= 2\frac{R}{b}(1+\varepsilon_i) - \frac{b}{R}\left(\frac{R}{b}\right)^2(1+\varepsilon_i)^2$$

$$= \frac{R}{b}\left[(2+2\varepsilon_i) - (1+2\varepsilon_i+\varepsilon_i^2)\right]$$

$$= \frac{R}{b}(1-\varepsilon_i^2) = \frac{R}{b}(1+\varepsilon_{i+1})$$

where $\varepsilon_{i+1} = \underline{-\varepsilon_i^2}$

Quadratic convergence ; # digits doubles at each step

$\therefore$ Complexity of division = complexity of multiplication

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TERMINATION

Iteration: $\quad x_{i+1} = \left\lfloor \dfrac{x_i + \lfloor a/x_i \rfloor}{2} \right\rfloor$

do floors hurt? Does program terminate?

Iteration is $\quad x_{i+1} = \dfrac{x_i + \frac{a}{x_i} - \alpha}{2} - \beta$

$$= \frac{x_i + \frac{a}{x_i}}{2} - \gamma$$

where $\gamma = \frac{\alpha}{2} + \beta$, $0 \le \gamma < 1$

Since $\frac{a+b}{2} \geq \sqrt{ab}$, $\quad \frac{x_i + \frac{a}{x_i}}{2} \geq \sqrt{a}$

So subtracting γ always leaves us $\geq \lfloor \sqrt{a} \rfloor$

won't stay stuck above if $\varepsilon_i < 1$ (good initial guess)