Review: high precision arithmetic
Multiplication

Division
Algorithm
Error Analysis
Termination

Review

Want millionth digit of $\sqrt{2}$

\[
\left\lfloor \sqrt{2 \cdot 10^{2d}} \right\rfloor \quad d = 106
\]

Compute $\left\lfloor \sqrt{a} \right\rfloor$ via Newton's method

$x_0 = 1$ (initial guess)

\[
x_{i+1} = \frac{x_i + a/x_i}{2}
\]

division!

Converges quadratically; $\pm$ correct digits doubles each step.
**Multiplication**

1. Naive Divide & Conquer method: \( \Theta(d^2) \) time
2. Karatsuba: \( \Theta(d \log_2^3) = \Theta(d^{1.584...}) \)
3. Toom-Cook generalizes Karatsuba (break into k=2 parts)
   \[
   T(d) = 5T(d/3) + \Theta(d) = \Theta(d^{1.465...})
   \]
4. Schonhage-Strassen - almost linear!
   \( \Theta(d \log d \log \log d) \) using FFT.

All of these are in `gmpy` package.

5. Fürer (2007): \( \Theta(n \log n 2^{O((\log^*n))} \) where \( \log^* n \) is iterated logarithm. # times log needs to be applied to get a number that is less than or equal to 1.

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**High-Precision Division**

We want high-precision rep of \( \frac{a}{b} \)

- Compute high-precision rep of \( \frac{1}{b} \) first
- High-precision rep of \( \frac{1}{b} \) means \( \lfloor \frac{r}{b} \rfloor \)

where \( r \) is large value s.t. it is easy to divide by \( r \).

Ex: \( r = 2^k \) for binary representations
Division

Newton's method for computing $\frac{R}{b}$

$$f(x) = \frac{1}{x} - \frac{b}{R} \quad \text{(zero at } x = \frac{R}{b})$$

$$f'(x) = -\frac{1}{x^2}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(\frac{1}{x_i} - \frac{b}{R}\right)}{-\frac{1}{x_i^2}}$$

$$x_{i+1} = x_i + x_i^2 \left(\frac{1}{x_i} - \frac{b}{R}\right) \quad \text{easy div multiply}$$

**Example**

Want $\frac{R}{b} = \frac{2^{16}}{5} = \frac{65536}{5} = 13107.2$

Try initial guess $\frac{2^{16}}{4} = 2^{14}$

$$x_0 = 2^{14} = 16384$$

$$x_1 = 2 \cdot (16384) - 5 \left(16384\right)^2 / 65536 = 12288$$

$$x_2 = 2 \cdot (12288) - 5 \left(12288\right)^2 / 65536 = 13056$$

$$x_3 = 2 \cdot (13056) - 5 \left(13056\right)^2 / 65536 = 13107$$
**Error Analysis**

\[ x_{i+1} = 2x_i - \frac{b}{R}x_i^2 \quad \text{Assume } x_i = \frac{R}{b}(1+\varepsilon_i) \]

\[ = 2 \frac{R}{b}(1+\varepsilon_i) - \frac{b}{R} \left( \frac{R}{b} \right)^2 (1+\varepsilon_i)^2 \]

\[ = \frac{R}{b} \left[ (2+2\varepsilon_i) - (1+2\varepsilon_i +\varepsilon_i^2) \right] \]

\[ = \frac{R}{b} (1-\varepsilon_i^2) = \frac{R}{b} (1+\varepsilon_{i+1}) \]

Where \( \varepsilon_{i+1} = -\varepsilon_i^2 \)

Quadratic convergence; # digits doubles at each step

\[ \text{Complexity of division} = \text{Complexity of multiplication} \]

**Termination**

Iteration: \( x_{i+1} = \left[ \frac{x_i + \sqrt{a/x_i}}{2} \right] \)

Do floors hurt? Does program terminate?

Iteration is \( x_{i+1} = x_i + \frac{a}{x_i} - \delta \)

\[ = x_i + \frac{a}{x_i} - \delta \quad \text{where} \quad \delta = \frac{a+b}{2} \quad 0 \leq \delta < 1 \]

Since \( a+b > \sqrt{ab} \), \( x_i + \frac{a}{x_i} > \sqrt{a} \)

so subtracting \( \delta \) always leaves us \( \geq \sqrt{a} \) (good initial guess)