Numerics I

Irrationals

Newton's method ($\sqrt{a}$, $1/b$)

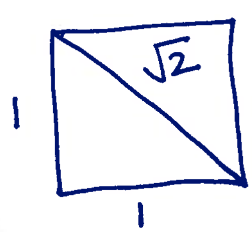
High precision multiply ←

" radix conversion (printing) } next time

" division

Pythagoras discovered that a square's diagonal and its side are incommensurable, i.e., could not be expressed as a ratio

— He called the ratio "speechless"!



Pythagoreans worshipped numbers

"All is number"

Irrationals were a threat!

Are there hidden patterns in irrationals?

$$\sqrt{2} = 1.414\ 213\ 562\ 373\ 095\ 048\ 801\ 688\ 724\ 209\ 698\ 078\ 569\ 671\ 875$$

Can you see a pattern?

DIGRESSION

Catalan numbers:

Set P of balanced parentheses strings are recursively defined as

1. $\lambda \in P$ (λ is empty string)
2. If $\alpha, \beta \in P$, then $(\alpha)\beta \in P$

Every nonempty balanced paren string can be obtained via Rule 2 from a unique α, β pair.

For example, (())()() obtained by $(\underbrace{()}_{\alpha})\underbrace{()()}_{\beta}$

Enumeration

C_n : number of balanced parentheses strings with exactly n pairs of parentheses

$C_0 = 1$ empty string

C_{n+1} ? Every string with $n+1$ pairs of parentheses can be obtained in a unique way via rule 2.

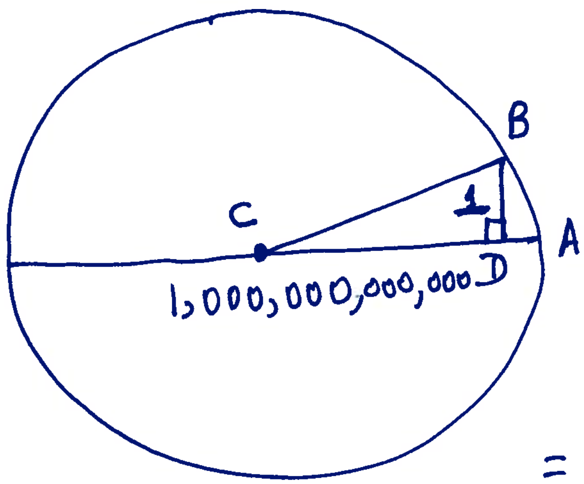
One paren pair comes explicitly from the rule.
k pairs from α, $n-k$ pairs from β

$$C_{n+1} = \sum_{k=0}^{n} C_k \cdot C_{n-k} \qquad n \geq 0$$

$C_0 = 1$ $\quad C_1 = C_0^2 = 1$ $\quad C_2 = C_0 C_1 + C_1 C_0 = 2$ $\quad C_3 = \cdots = 5$

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, . . .

GEOMETRY PROBLEM



BD = 1

What is AD?

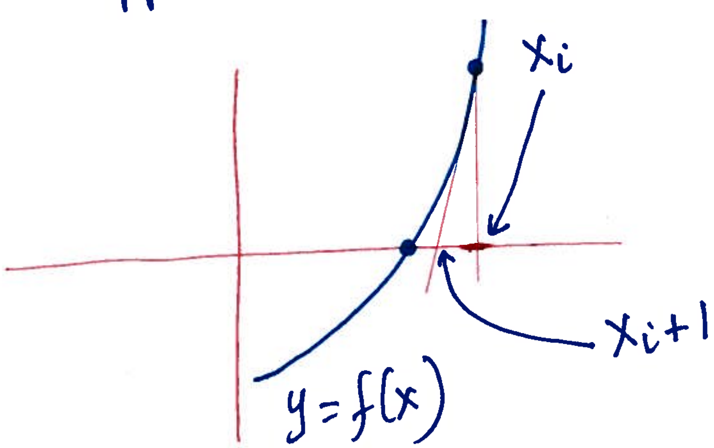
$$AD = AC - CD$$

$$= 500{,}000{,}000{,}000 - \sqrt{\underbrace{500{,}000{,}000{,}000^2 - 1}_{a}}$$

let's calculate AD to a million places!

Newton's Method.

Find root of $f(x) = 0$ through successive approximation. e.g., $f(x) = x^2 - a$



Tangent at $(x_i, f(x_i))$ is line $y = f(x_i) + \underbrace{f'(x_i)}_{\text{derivative}} \cdot (x - x_i)$

x_{i+1} = intercept on x-axis

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Square roots

$$f(x) = x^2 - a$$

$$x_{i+1} = x_i - \frac{(x_i^2 - a)}{2x_i} = \frac{x_i + \frac{a}{x_i}}{2}$$

Ex: $a = 2$

$x_0 = 1.000\ 000\ 000$
$x_1 = 1.500\ 000\ 000$
$x_2 = 1.416\ 666\ 666$
$x_3 = 1.414\ 215\ 686$
$x_4 = 1.414\ 213\ 562$

quadratic convergence, # digits doubles

High precision computation

$\sqrt{2}$ to d-digit precision : $\underbrace{1.414213562373}_{d \text{ digits}}\cdots$

Want integer $\lfloor 10^d \sqrt{2} \rfloor$

$= \lfloor \sqrt{2 \cdot 10^{2d}} \rfloor$ integral part of square root

Can still use Newton's Method.

Let's try it on $\sqrt{2}$, and our segment AD!

See anything interesting?

High-Precision Multiplication

Multiplying two n-digit numbers (radix $r = 2, 10$)

$$0 \le x, y < r^n$$

x_1 = high half
x_0 = low half

$$x = x_1 \cdot r^{n/2} + x_0$$
$$y = y_1 \cdot r^{n/2} + y_0$$

$$0 \le x_0, x_1 < r^{n/2}$$
$$0 \le y_0, y_1 < r^{n/2}$$

$$z = x \cdot y = \underline{x_1 y_1} \cdot r^n + (\underline{x_0 \cdot y_1} + \underline{x_1 \cdot y_0})\, r^{n/2} + \underline{x_0 \cdot y_0}$$

4 multiplications of half-sized #'s $\Rightarrow$ quadratic algorithm $\Theta(n^2)$ time

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Karatsuba's Method

$\log_2 n$

$4T(n/2)$

$4^{\log_2 n} = n^{\log_2 4} = n^2$

$\log_2 n$

$3T(n/2)$

$3^{\log_2 n} = n^{\log_2 3}$

Let $z_0 = \underline{x_0 \cdot y_0}$

$z_2 = \underline{x_2 \cdot y_2}$

$z_1 = \underline{(x_0 + x_1) \cdot (y_0 + y_1)} - z_0 - z_2$

$= x_0 y_1 + x_1 y_0$

3 multiplies

$$z = z_2 \cdot r^n + z_1 \cdot r^{n/2} + z_0$$

$T(n)$ = time to multiply two n-digit #'s

$= 3T(n/2) + \Theta(n)$

$= \Theta(n^{\log_2 3}) = \Theta(n^{1.5849625\cdots})$

better than $\Theta(n^2)$. Python does this.

Error Analysis of Newton's Method

Suppose $x_n = \sqrt{a} \cdot (1 + \varepsilon_n)$ $\qquad$ ε_n may be $+$ or $-$

Then
$$x_{n+1} = \frac{x_n + a/x_n}{2}$$
$$= \frac{\sqrt{a}(1+\varepsilon_n) + \frac{a}{\sqrt{a}(1+\varepsilon_n)}}{2}$$
$$= \sqrt{a}\ \frac{\left((1+\varepsilon_n) + \frac{1}{(1+\varepsilon_n)}\right)}{2}$$
$$= \sqrt{a}\left(\frac{2 + 2\varepsilon_n + \varepsilon_n^2}{2(1+\varepsilon_n)}\right)$$
$$= \sqrt{a}\left(1 + \frac{\varepsilon_n^2}{2(1+\varepsilon_n)}\right)$$

$$\therefore \quad \varepsilon_{n+1} = \frac{\varepsilon_n^2}{2(1+\varepsilon_n)}$$

quadratic convergence, as #digits doubles.