Numerics I

Irrationals
Newton's method \((\sqrt{a}, \frac{1}{b})\)
High precision multiply \(\leadsto\) radix conversion (printing)
""
""
next time

Pythagoras discovered that a square's diagonal and its side are incommensurable, i.e., could not be expressed as a ratio.
- He called the ratio "speechless"!

\[ \begin{array}{c}
\sqrt{2} \\
1 \\
1 \\
\end{array} \]

Pythagoreans worshipped numbers
"All is number"
Irrationals were a threat!

Are there hidden patterns in irrationals?
\[
\sqrt{2} = 1.414213562373095048801688724209698078569671875
\]

Can you see a pattern?

---

**DIGRESSION**

Catalan numbers:

Set \( P \) of balanced parentheses strings are recursively defined as:

1. \( \lambda \in P \) (\( \lambda \) is empty string)
2. If \( \alpha, \beta \in P \), then \( \alpha\beta \in P \)

Every nonempty balanced parenthesis string can be obtained via Rule 2 from a unique \( \alpha, \beta \) pair.

For example, \((())(())\) obtained by \(\frac{4}{2} \cdot \frac{2}{2} \cdot \frac{2}{2} \).
Enumeration

\[ C_n : \text{number of balanced parentheses strings with exactly } n \text{ pairs of parentheses} \]

\[ C_0 = 1 \quad \text{empty string} \]

\[ C_{n+1} \text{? Every string with } n+1 \text{ pairs of parentheses can be obtained in a unique way via rule 2.} \]

One paren pair comes explicitly from the rule. \( k \) pairs from \( A \), \( n-k \) pairs from \( B \)

\[ C_{n+1} = \sum_{k=0}^{n} C_k \cdot C_{n-k} \quad n \geq 0 \]

\[ C_0 = 1 \quad C_1 = C_0^2 \quad C_2 = C_0 C_1 + C_1 C_0 \quad C_3 = \ldots \]
\[ = 1 \quad = 2 \quad = 5 \]

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, \ldots
Geometry Problem

BD = 1
What is AD?
AD = AC - CD
= \frac{500,000,000,000,000}{1,000,000,000,000} - \sqrt{\frac{500,000,000,000,000^2 - 1}{1,000,000,000,000}}

Let's calculate AD to a million places!

Newton's Method

Find root of \( f(x) = 0 \) through successive approximation. E.g., \( f(x) = x^2 - a \)

Tangent at \((x_i, f(x_i))\) is line \( y = f(x_i) + \frac{f'(x_i)}{f'(x_i)} \cdot (x - x_i) \)

\( x_{i+1} = \text{intercept on x-axis} \)
\( x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \)
Square roots

\[ f(x) = x^2 - a \]

\[ x_{i+1} = x_i - \frac{(x_i^2 - a)}{2x_i} = x_i + \frac{a}{x_i} \]

Ex: \[ x_0 = 1.00000000 \quad a = 2 \]
\[ x_1 = 1.50000000 \]
\[ x_2 = 1.41666666 \]
\[ x_3 = 1.414215686 \]
\[ x_4 = 1.414213562 \]

Quadratic convergence, # digits doubles

High precision computation

\[ \sqrt{2} \text{ to } d \text{-digit precision : } 1.414213562 \ldots \]

Want integer \[ \left\lfloor 10^d \sqrt{2} \right\rfloor \]

\[ = \left\lfloor \sqrt{2.1024} \right\rfloor \]

Integral part of square root

Can still use Newton's Method.

Let's try it on \(\sqrt{2}\), and our segment \(AB\)!

See anything interesting?
High-Precision Multiplication

Multiplying two n-digit numbers (radix $r = 2, 10$)

$0 \leq x, y \leq r^n$

$x = x_1 \cdot r^{n/2} + x_0$
$y = y_1 \cdot r^{n/2} + y_0$

$x_1 = \text{high half}$
$x_0 = \text{low half}$
$0 \leq x_0, x_1 < r^{n/2}$
$0 \leq y_0, y_1 < r^{n/2}$

$z = x \cdot y = x_1 y_1 \cdot r^n + (x_0 y_1 + x_1 y_0) r^{n/2} + x_0 y_0$

4 multiplications of half-sized #’s $\Rightarrow$ quadratic algorithm $\Theta(n^2)$ time

Karat Subba’s Method

Let $z_0 = x_0 \cdot y_0$
$z_2 = x_2 \cdot y_2$
$z_1 = (x_0 + x_1) \cdot (y_0 + y_1) - z_0 - z_2$

$= x_0 y_1 + x_1 y_0$

$z = z_2 \cdot r^n + z \cdot r^{n/2} + z_0$

$T(n) = \text{time to multiply two n-digit #’s}$

$= 3T(n/2) + \Theta(n)$

$= \Theta(n^{\log_2 3}) = \Theta(n^{1.5849625...})$

better than $\Theta(n^2)$. Python does this.
Error Analysis of Newton's Method

Suppose \( x_n = \sqrt{a} \cdot (1 + \varepsilon_n) \), \( \varepsilon_n \) may be \( + \) or \( - \).

Then \( x_{n+1} = \frac{x_n + a/x_n}{2} \)

\[
\begin{align*}
&= \frac{\sqrt{a} \cdot (1 + \varepsilon_n) + a}{\sqrt{a} \cdot (1 + \varepsilon_n)} \\
&= \sqrt{a} \cdot \frac{2}{(1 + \varepsilon_n) + 1} \\
&= \sqrt{a} \cdot \left(1 + \frac{\varepsilon_n^2}{2(1 + \varepsilon_n)}\right)
\end{align*}
\]

\[\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2(1 + \varepsilon_n)}\]

Quadratic convergence, as \#digits doubles.