6.006
Spring 2008
L18

Shortest Paths IV: Speeding Up Dijkstra

Single-source single-target Dijkstra

Bidirectional search

Goal directed search

- Potentials
- Landmarks

Reading: Wagner paper on website (upto Sect. 3.2)

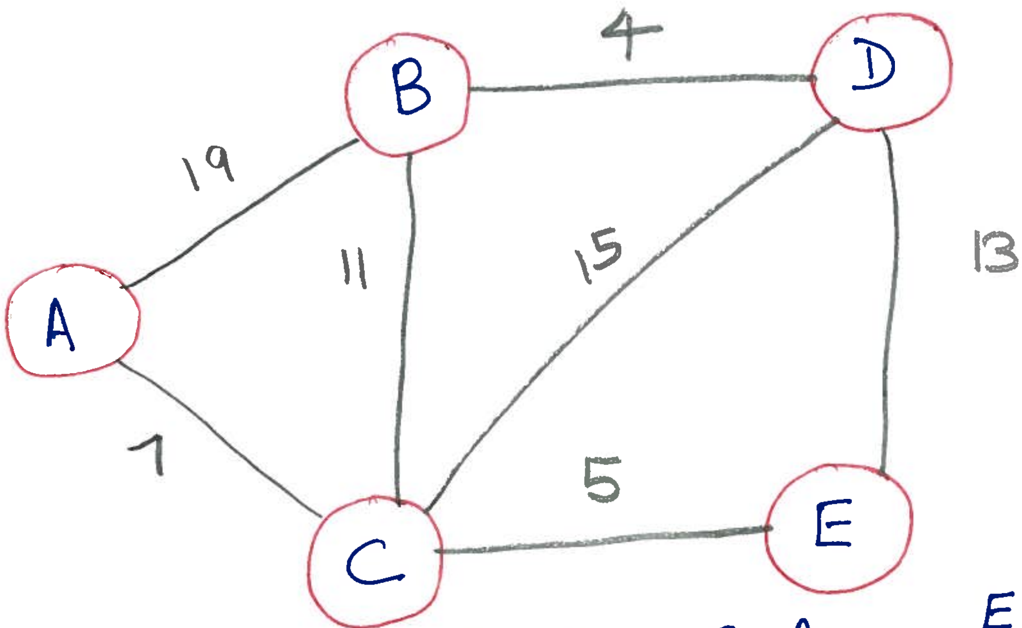
DIJKSTRA single-source, single-target

```
Initialize()
  ..
Q ← V[G]
while Q ≠ ∅
      do u ← EXTRACT-MIN (Q)
         for each vertex v ∈ Adj[u]
             do RELAX (u, v, w)
```

if $u=t$, stop!

Observation: If only shortest path from s to t is required, stop when t is removed from Q, i.e. when $u=t$

DIJKSTRA DEMO



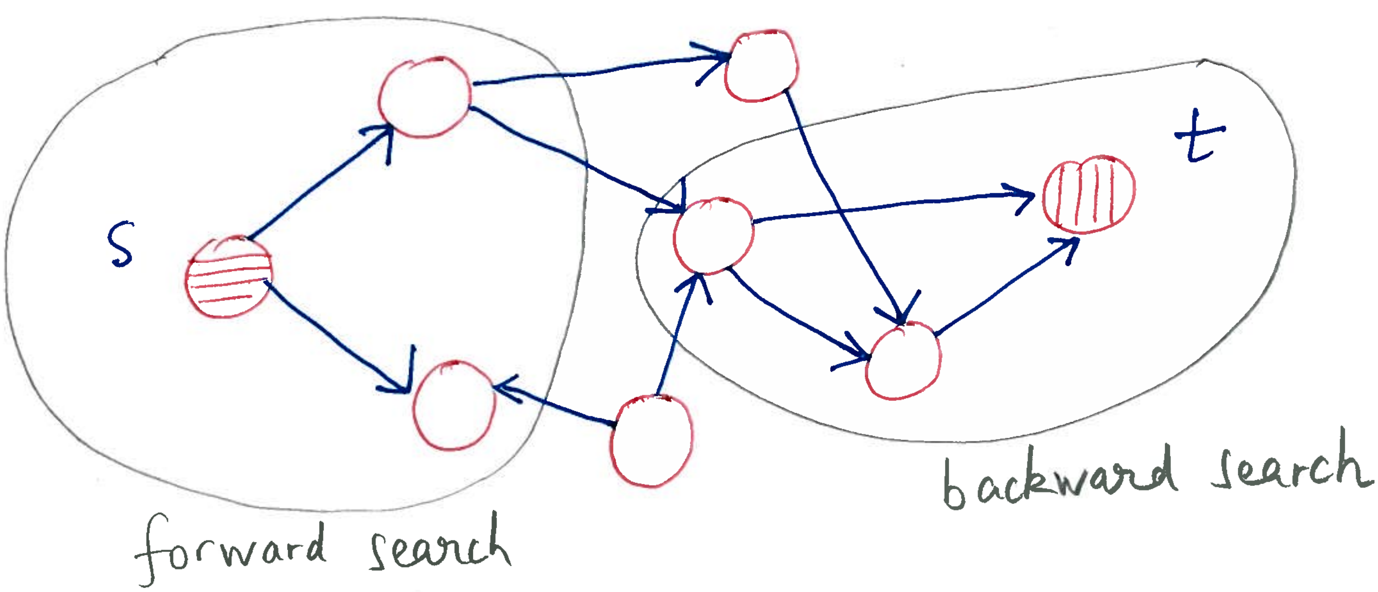
A	C	E	B	D
	7	12	18	22

D	B	E	C	A
	4	13	15	22

E	C	A	D	B
	5	12	13	16

Bi-Directional Search

Note: Speedup techniques covered here do not change worst-case behavior, but reduce the number of visited vertices in practice.



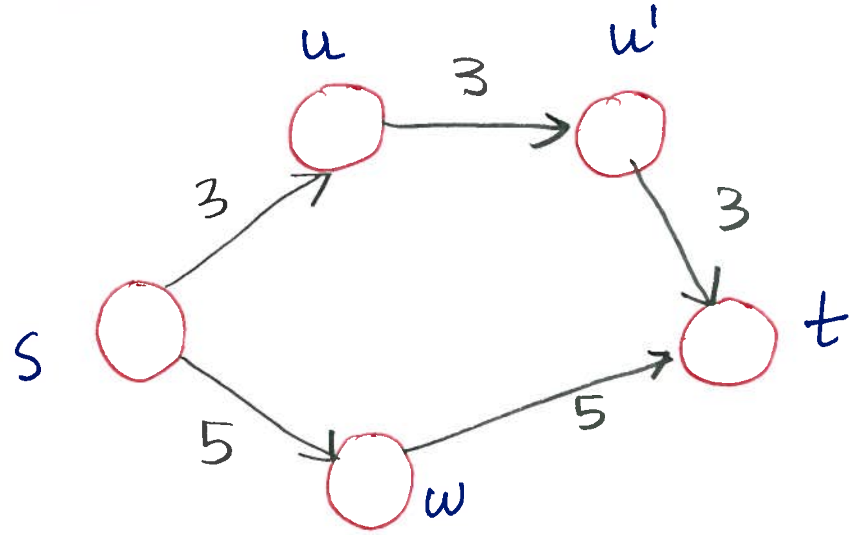
Bi-D Search

Alternate forward search from s
backward search from t
(follow edges backward)

$d_f(u)$ distances for forward search
$d_b(u)$ " " backward "

Algorithm terminates when some vertex w has been processed, i.e., deleted from the queue of both searches, Q_f and Q_b.

Bi-D Search

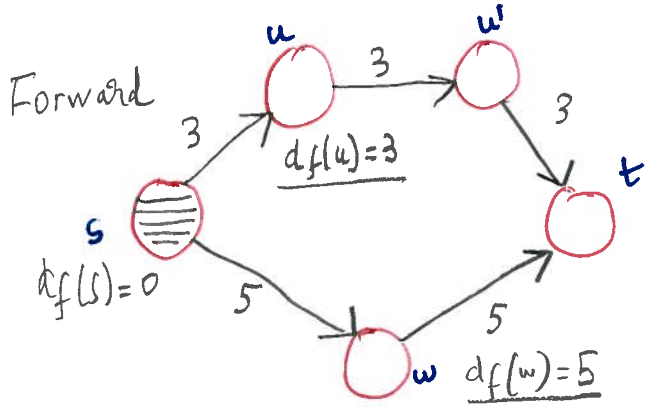


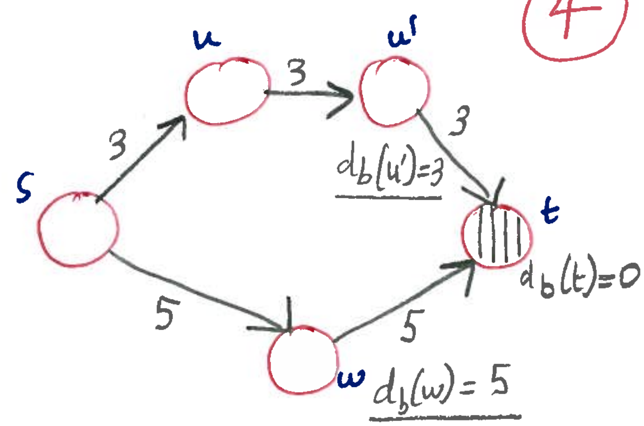
Subtlety: After search terminates, find node **x** with minimum value of $d_f(x) + d_b(x)$.

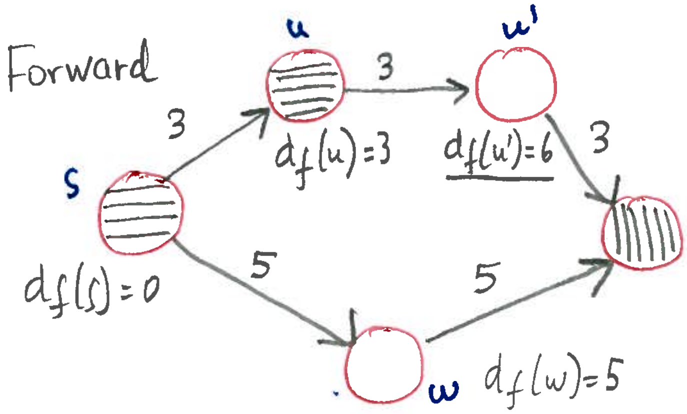
X may not be the vertex w that caused termination as in example to the left!

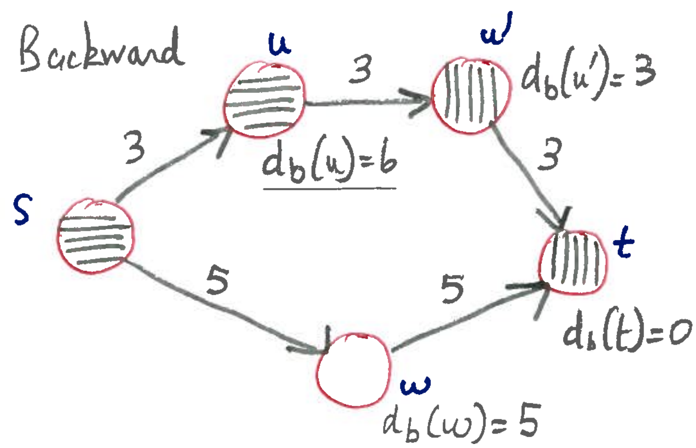
Find shortest path from s to **x** using Π_f and shortest path backwards from t to X using Π_b.

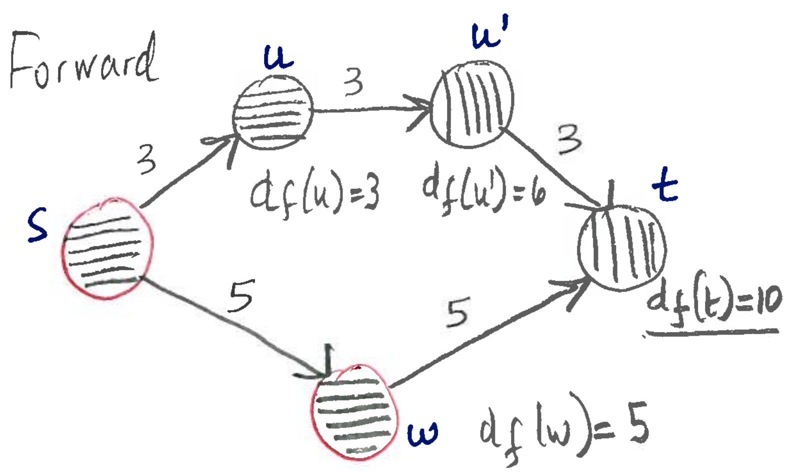
Note: x will have been deleted from either Q_f or Q_b or both.

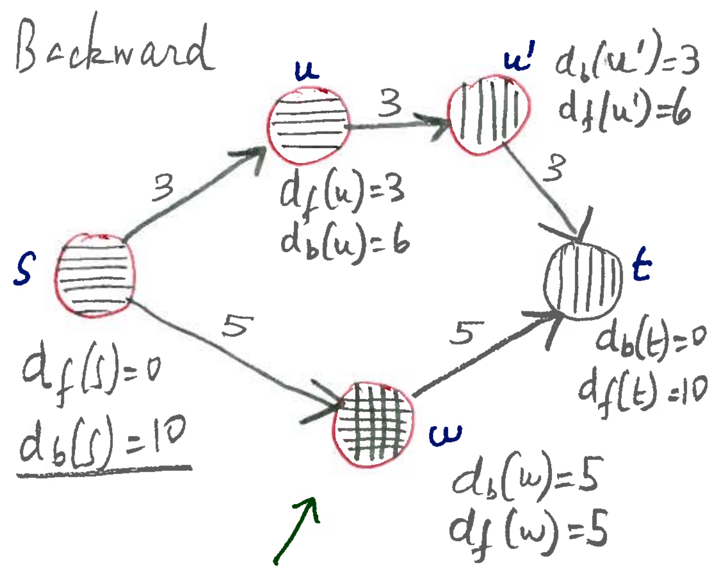












deleted from both queues,
so terminate!

Min value for $d_f(x) + d_b(x)$ over all vertices that have been processed in at least one search

$d_f(u) + d_b(u) = 3 + 6 = 9 \qquad d_f(u') + d_b(u') = 6+3 = 9$

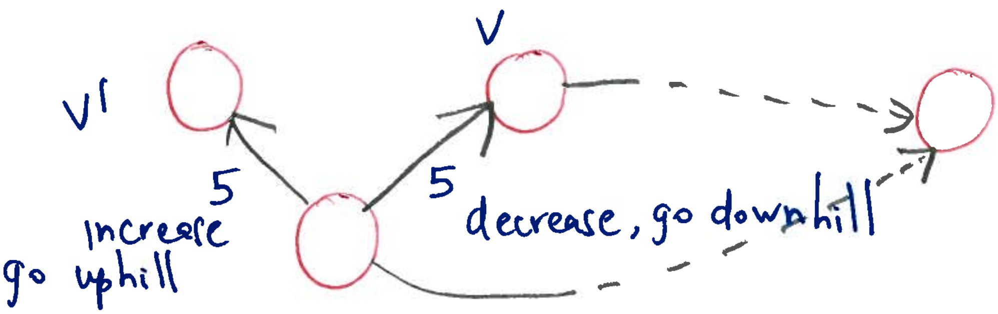
$d_f(w) + d_b(w) = 5+5 = 10$.

Goal-Directed Search or A^*

Modify edge weights with potential function over vertices

$$\overline{w}(u,v) = w(u,v) - \lambda_t(u) + \lambda_t(v)$$

Search toward target:



Correctness

$$\overline{w}(p) = w(p) - \lambda_t(s) + \lambda_t(t)$$

$$s \overset{p}{\rightsquigarrow} t \qquad s \underset{p'}{\rightsquigarrow} t$$

So shortest paths are maintained in modified graph with $\overline{w}$ weights.

To apply Dijkstra, we need $\overline{w}(u,v) \geq 0$ for all (u,v).

Choose potential function appropriately, to be feasible.

LANDMARKS.

Small set of landmarks $L \subset V$

For all $u \in V$, $\ell \in L$, precompute $\delta(u, \ell)$

Potential $\lambda_t^{(\ell)}(u) = \delta(u, \ell) - \delta(t, \ell)$ for each ℓ

CLAIM: $\lambda_t^{(\ell)}$ is feasible.

FEASIBILITY

$$\bar{w}(u,v) = w(u,v) - \lambda_t^{(\ell)}(u) + \lambda_t^{(\ell)}(v)$$

$$= w(u,v) - \delta(u,\ell) + \delta(t,\ell) + \delta(v,\ell) - \delta(t,\ell)$$

$$= w(u,v) - \delta(u,\ell) + \delta(v,\ell) \geq 0$$

by the Δ-inequality

$$\lambda_t(u) = \max_{\ell \in L} \lambda_t^{(\ell)}(u)$$ is also feasible