6.006 L16
Spring 2008

Shortest Paths: Bellman Ford

Review: Notation
Generic s.p. algorithm
Bellman Ford algorithm
- analysis
- correctness

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path $p = \langle v_0, v_1, \ldots v_k \rangle$

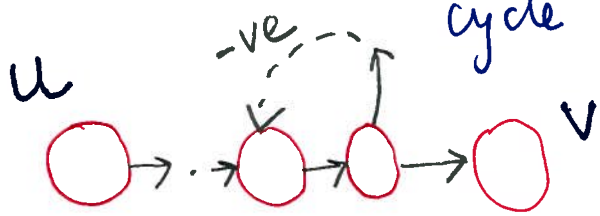
$(v_i, v_{i+1}) \in E \quad 0 \leq i < k$

$$w(p) = \sum_{i=0}^{k-1} w(v_i, v_{i+1})$$

Shortest path weight from u to v is $\delta(u,v)$

$\delta(u,v)$ is ∞ if v is unreachable from u

undefined if there is a negative cycle on some path from u to v



General Structure of S.P. Algs

Initialize : for $v \in V$: $d[v] \leftarrow \infty$
$\pi[v] \leftarrow$ NIL

$d[s] \leftarrow 0$

Main : repeat :
Select edge (u,v) [somehow]

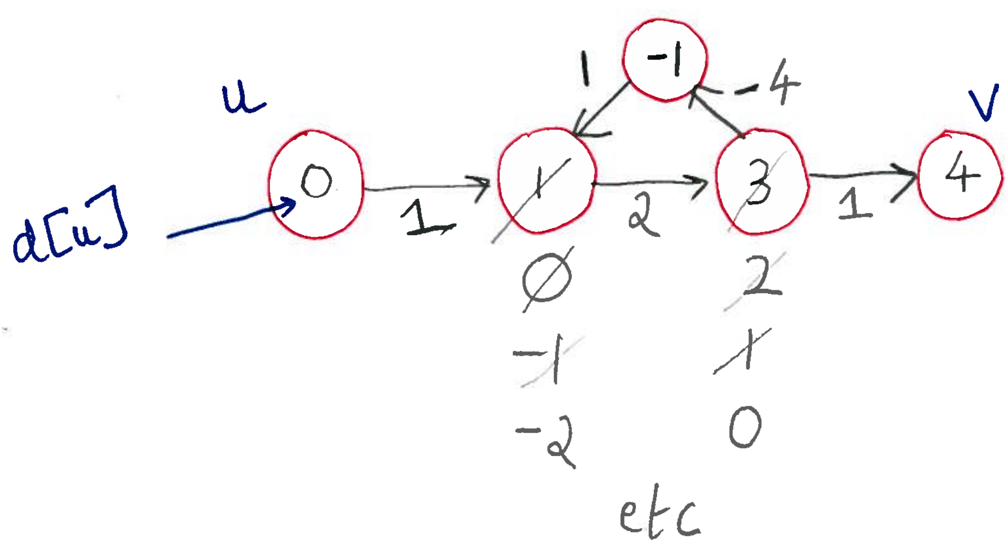
"Relax" edge (u,v)

if $d[v] > d[u] + w(u,v)$:
  $d[v] \leftarrow d[u] + w(u,v)$
  $\pi[v] \leftarrow u$

until you can't relax any more edges
or you're tired or ...

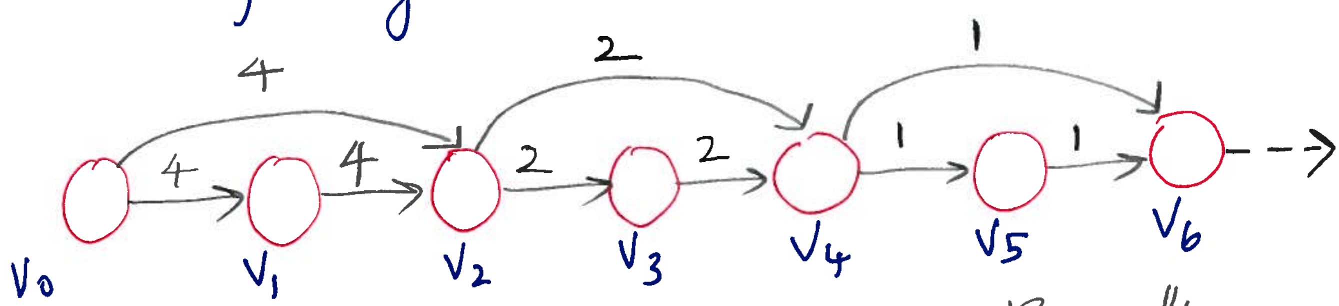
COMPLEXITY

Termination: Algorithm will continually relax edges when there are negative cycles present.



COMPLEXITY

Could be exponential time with poor choice of edges.



V_1	V_2	V_3	V_4	V_5	V_6
4	8	10	12	13	14
					13
			10	11	12
	4	6	8	9	10

$T(n) = 3 + 2T(n-2)$

$T(n) = \theta(2^{n/2})$

ORDER
(V_0, V_1)
(V_1, V_2)
all of $V_2 .. V_n$
(V_0, V_2)
all of $V_2 .. V_n$

5-MINUTE 6.006

Here's what I want you to remember from 6.006 five years after you graduate.

Exponential bad, Polynomial good.

$$T(n) = C_1 + C_2 T(n - C_3)$$

if $C_2 > 1$, trouble!

Divide & Explode

$$T(n) = C_1 + C_2 T\left(\frac{n}{C_3}\right)$$

$C_2 > 1$ okay if $C_3 > 1$

Divide & Conquer

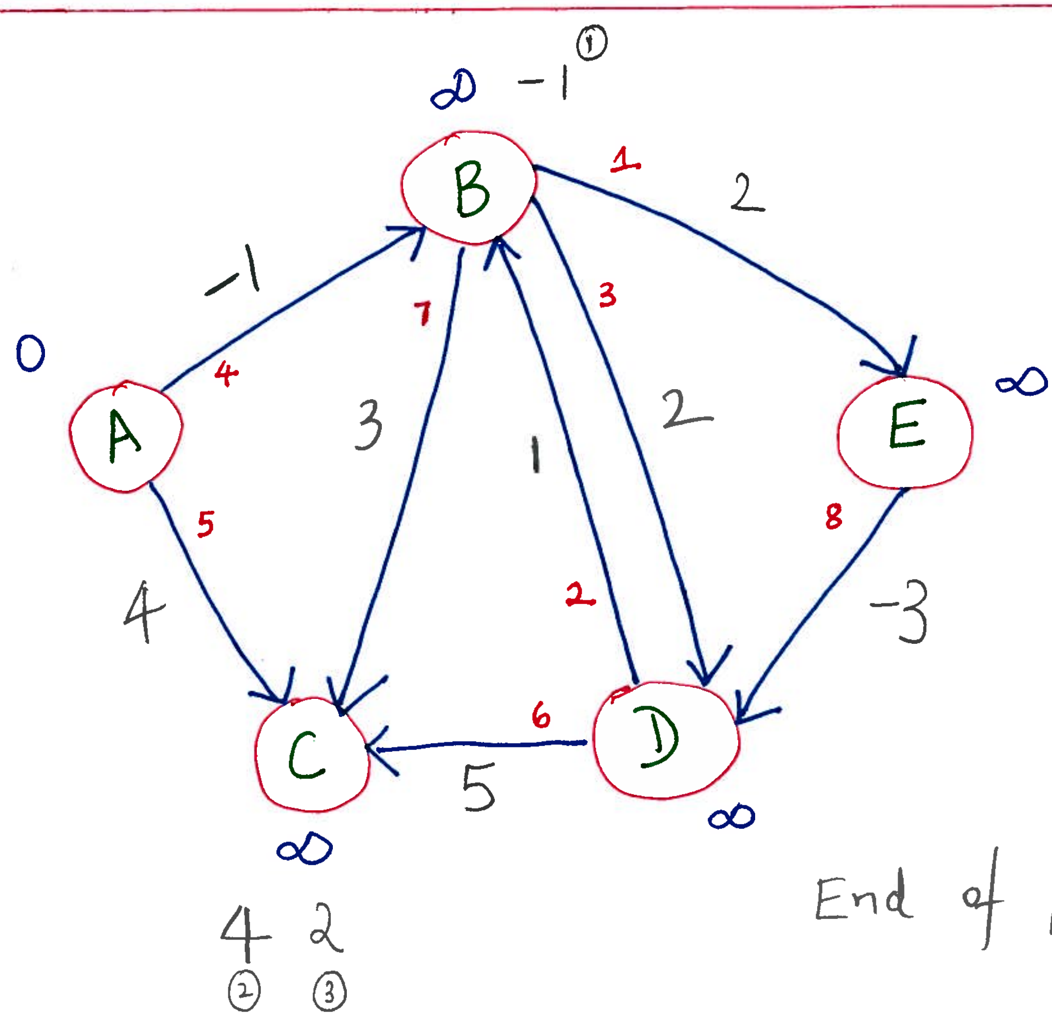
BELLMAN—FORD (G, w, s)

Initialize()
for $i = 1$ to $|V| - 1$
    for each edge $(u, v) \in E$:
        Relax(u, v)

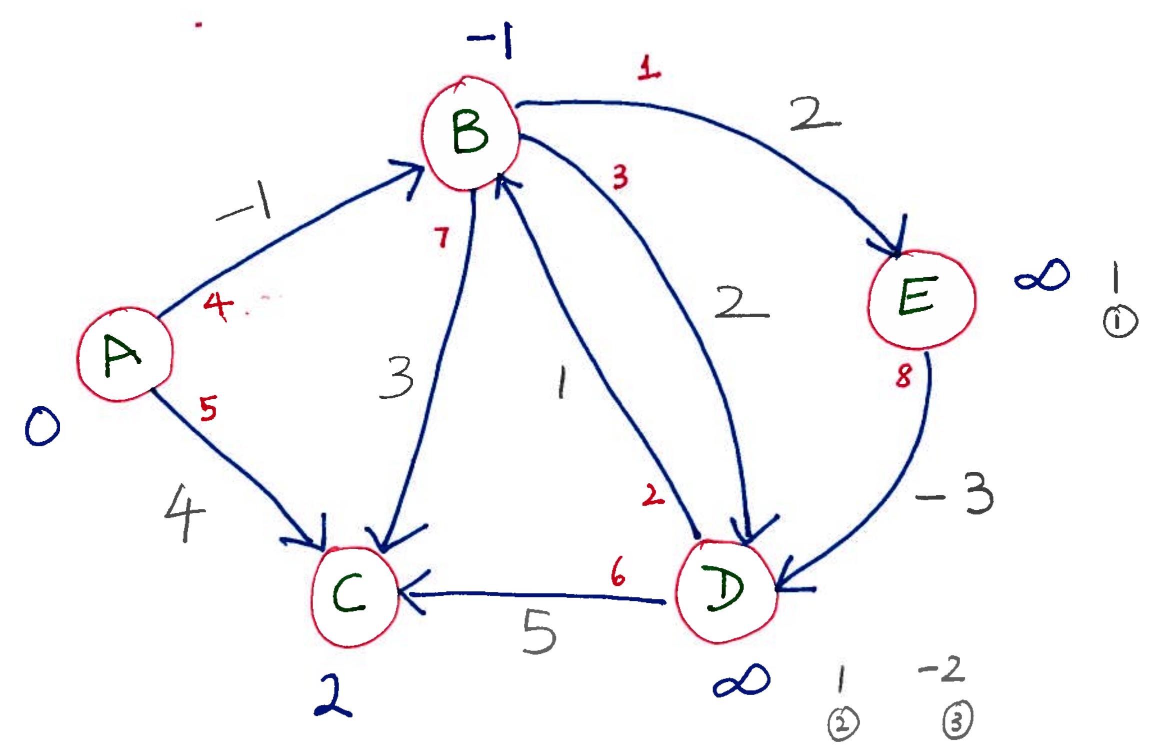
$O(VE)$

for each edge $(u, v) \in E$
    do if $d[v] > d[u] + w(u, v)$
        then report a negative-weight cycle exists

At the end, $d[v] = \delta(s, v)$, if no negative-weight cycles



End of pass 1

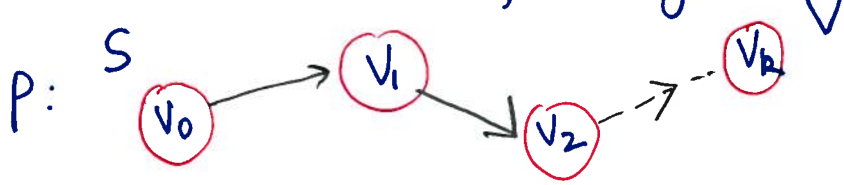


End of pass 2 (and 3 and 4)

Theorem

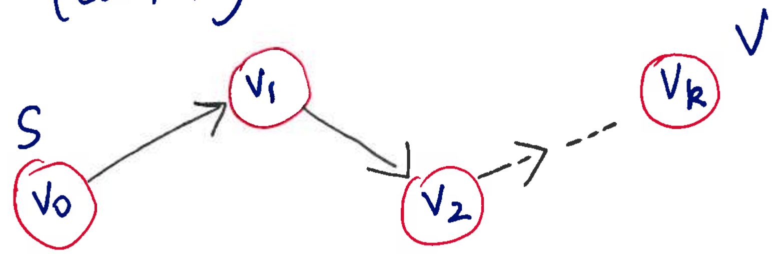
If $G = (V, E)$ contains no negative weight cycles, then after Bellman-Ford executes $d[v] = \delta(s, v)$ for all $v \in V$.

Proof: $v \in V$ be any vertex.
Consider path p from s to v that is a shortest path with minimum number of edges.



$$\delta(s, v_i) = \delta(s, v_{i-1}) + w(v_{i-1}, v_i)$$

Proof (contd.):

p: 

Initially $d[v_0] = 0 = \delta(s, v_0)$ and is unchanged since no negative cycles

After 1 pass through E, we have $d[v_1] = \delta(s, v_1)$

" 2 passes " " $d[v_2] = \delta(s, v_2)$

. . .

" k passes " $d[v_k] = \delta(s, v_k)$

No negative-wt cycles $\Rightarrow$ p is simple $\Rightarrow$ p has $\leq |V|-1$ edges $\boxtimes$

COROLLARY

If a value $d[v]$ fails to converge after $|V|-1$ passes, there exists a negative-weight cycle reachable from s.