6.006 Recitation
Build 2008.16
Coming up next...

• Sorting
  • Scenic Tour: Insertion Sort, Selection Sort, Merge Sort
  • New Kid on the Block: Merge Sort
• Priority Queues
  • Heap-Based Implementation
Sorting

• Input: array $a$ of $N$ keys

• Output: a permutation $a_s$ of $a$ such that $a_s[k] < a_s[k+1]$ 

• Stable sorting:
Sorting

- Maybe the oldest problem in CS
- Reflects our growing understanding of algorithm and data structures
- Who gives a damn?
- All those database tools out there
# Sorting Algorithms: Criteria

<table>
<thead>
<tr>
<th>What</th>
<th>Why</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed</td>
<td>That’s what 6.006 is about</td>
</tr>
<tr>
<td>Auxiliary Memory</td>
<td>External sorting, memory isn’t that cheap</td>
</tr>
<tr>
<td>Simple Method</td>
<td>You’re learning / coding / debugging / analyzing it</td>
</tr>
<tr>
<td># comparisons, data moving</td>
<td>Keys may be large (strings) or slow to move (flash memory)</td>
</tr>
</tbody>
</table>
Insertion Sort

- Base: $a[0:1]$ has 1 element $\Rightarrow$ is sorted

- Induction: $a[0:k]$ is sorted, want to grow to $a[0:k+1]$ sorted
  
  - find position of $a[k+1]$ in $a[0:k]$
  
  - insert $a[k+1]$ in $a[0:k]$
Insertion Sort: Costs

- Find position for $a[k+1]$ in $a[0:k]$ - $O(\log(k))$
  - use binary search
- Insert $a[k+1]$ in $a[0:k]$: $O(k)$
  - shift elements
- Total cost: $O(N \cdot \log(N)) + O(N^2) = O(N^2)$

Pros:
- Optimal number of comparisons
- $O(1)$ extra memory (no auxiliary arrays)

Cons:
- Moves elements around a lot
Selection Sort

- Base case: $a[0:0]$ has the smallest 0 elements in $a$
- Induction: $a[0:k]$ has the smallest $k$ elements in $a$, sorted; want to expand to $a[k+1]$
  - find min($a[k+1:N]$)
  - swap it with $a[k+1]$

```
Initial: 5 8 2 7 1 4 3 6
           1 8 2 7 5 4 3 6
           1 2 8 7 5 4 3 6
           1 2 3 7 5 4 8 6
           1 2 3 4 5 7 8 6
           1 2 3 4 5 7 8 6
           1 2 3 4 5 6 8 7
           1 2 3 4 5 6 7 8
```
**Selection Sort: Costs**

- find minimum in \(a[k+1:N]\) - \(O(N-k)\)
  - scan every element
- swap with \(a[k]\) - \(O(1)\)
  - need help for this?
- Total cost: \(O(N^2) + O(N) = O(N^2)\)

**Pros:**
- Optimal in terms of moving data around
- \(O(1)\) extra memory (no auxiliary arrays)

**Cons:**
- Compares a lot
Merge-Sort

1. Divide
   • Break into 2 sublists

2. Conquer
   • 1-elements lists are sorted

3. Profit
   • Merge sorted sublists

There is no step 6
There is no step 7
There is no step 8
Merge-Sort: Cost

- You should be ashamed of if you don’t know!

- \( T(N) = 2T(N/2) + \Theta(N) \)

- Recursion tree
  - \( O(\log(N)) \) levels, \( O(N) \) work / level

- Total cost: \( O(N \cdot \log(N)) \)

- Pros:
  - Optimal number of comparisons
  - Fast

- Cons:
  - \( O(N) \) extra memory (for merging)
BST Sort

- Build a BST out of the keys
- Use inorder traversal to obtain the keys in sorted order
- Or go to minimum(), then call successor() until it returns None
BST Sort: Cost

- Building the BST - $O(N \cdot \log(N))$
  - Use a balanced tree
- Traversing the BST - $O(N)$
  - Even if not balanced
- Total cost: $O(N \cdot \log(N))$

- Pros:
  - Fast (asymptotically)
- Cons:
  - Large constant
  - $O(N)$ extra memory (left/right pointers)
  - Complex code
# Best of Breed Sorting

<table>
<thead>
<tr>
<th>Feature</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Speed</strong></td>
<td>$O(N \cdot \log(N))$</td>
</tr>
<tr>
<td><strong>Auxiliary Memory</strong></td>
<td>$O(1)$</td>
</tr>
<tr>
<td><strong>Code complexity</strong></td>
<td>Simple</td>
</tr>
<tr>
<td><strong>Comparisons</strong></td>
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</tr>
<tr>
<td><strong>Data movement</strong></td>
<td>$O(N)$</td>
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</tbody>
</table>
Heap-Sort

<table>
<thead>
<tr>
<th></th>
<th>Speed</th>
<th>Auxiliary Memory</th>
<th>Code complexity</th>
<th>Comparisons</th>
<th>Data movement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$O(N \cdot \log(N))$</td>
<td>$O(1)$</td>
<td>Simple</td>
<td>$O(N \cdot \log(N))$</td>
<td>$O(N)$</td>
</tr>
<tr>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
</tr>
</tbody>
</table>
Heap-Sort uses a... Heap (creative, eh?)

- Max-Heap DT
  - Almost complete binary tree
  - Root node’s key $\geq$ its children’s keys
  - Subtrees rooted at children are Max-Heaps as well
Max-Heap Properties

- Very easy to find max. value
  - look at root, doh
- Unlike BSTs, it’s very hard to find any other value
  - 6 (3rd largest key) at same level as 1 (min. key)
Heaps Inside Arrays

• THIS IS WHY HEAPS ROCK OVER BSTs

• No need to store a heap as a binary tree (left, right, parent pointers)

• Store keys inside array, in level-order traversal
Heaps Inside Arrays

- Work with arrays, think in terms of trees
- Left subtree of 8 is in bold... pretty mind-boggling, eh?
- Prey that you don’t have to debug this
Heaps Inside Arrays

- root index: 1
- left_child(node_index):
  - node_index $\cdot 2$
- right_child(node_index):
  - node_index $\cdot 2 + 1$
- parent(node_index):
  - $\lfloor \frac{\text{node\_index}}{2} \rfloor$
Heaps Inside Arrays

- How to recall this
  1. draw the damn heap (see right)
  2. remember the concept (divide / multiply by 2)
  3. work it out with the drawing
Heaps Inside Arrays: Python Perspective

- Lists are the closest thing to an array.
- Except they grow.
  - Just like our growing hashes.
  - Amortized $O(1)$ per operation.

   1  2  3  4  5  6  7  8
   7  5  8  3  1  4  6  2
Messing with Heaps

- Goal:
  1. Change any key
  2. Restore Max-Heap invariants
Messing with Heaps: Percolate

- **Issue**
  - key’s node becomes smaller than children
  - only possible after decreasing a key

- **Solution**
  - percolate (huh??)
Messing with Heaps: Percolate

- Percolate:
  - swap node’s key with max(left child key, right child key)
  - Max-Heap restored locally
  - the child we didn’t touch still roots a Max-Heap
Messing with Heaps: Percolate

- Percolate
- Issue: swapping decreased the key of the child touched
  - child might not root a Max-Heap
- Solution: keep percolating
Messing with Heaps: Percolate

- Percolating is finite:
  - leaves are always Max-Heaps

- Percolate cost:
  - $O(\text{heap height} - \text{node's level})$
  - $O(\log(N) - \log(\text{node}))$
Messing with Heaps: Sift

- **Issue**
  - key’s node becomes larger than parent
  - only possible after increasing a key

- **Solution**
  - sift (huh??)
Messing with Heaps: Sift

- Sift
  - swap node’s key with parent’s key
  - parent’s key was $\geq$ node’s key, so must be $\geq$ children keys
  - Max-Heap restored for node’s subtree
Messing with Heaps: Sift

- Sift

- Issue: swapping increased the key of the parent
  - parent might not root a Max-Heap

- Solution: keep sifting
Messing with Heaps: Sift

- Sifting is finite:
  - root has no parent, so it can be increased at will

- Sift cost:
  - $O(\text{height})$
  - $O(\log(\text{node}))$
Messing with Heaps

- **Update**(node, new_key)
  - old_key ← heap[node].key
  - heap[node].key ← new_key
  - if new_key < old_key: **sift**(node)
  - else: **percolate**(node)
Messing with Heaps II

- Goal
  - Want to shrink or grow the heap

- Growing:
  - inserting keys

- Shrinking:
  - deleting keys
Messing with Heaps II: One More Node

- Can always insert $-\infty$ at the end of the heap
- Max-Heap will not be violated
- Can only add to the end, otherwise we wouldn’t get an (almost) complete binary tree
Messing with Heaps II: One More Node

- Insert any key
- Insert $-\infty$ at the end of the heap
- Change node’s key to desired key
- Sift
Messing with Heaps II: One More Node

- Insertion cost
  - insert \(-\infty\) at the end of the heap - \(O(1)\)
  - change node’s key to new key - \(O(1)\)
  - sift - \(O(\log(N))\)
- Total cost: \(O(\log(N))\)
Messing with Heaps II: One More Less Node

- Can always delete last node
- Max-Heap will not be violated
- It must be the last node, otherwise the binary tree won’t be (almost) complete
Messing with Heaps II: One More Less Node

- Deleting root
  - Replace root key with last key
- Delete last node
- Percolate
Messing with Heaps II: One More Less Node

- Deleting root cost
  - Replace root key with last key - $O(1)$
  - Delete last - $O(1)$
  - Percolate - $O(\log(N))$

- Total cost: $O(\log(N))$
Messing with Heaps II: One More Less Node

- Deleting any node
- Change key to $+\infty$
- Sift
- Delete root
Messing with Heaps II: One More Less Node

- Deletion cost
  - Change key to $+\infty$ - $O(1)$
  - Sift - $O(\log(N))$
  - Remove root - $O(\log(N))$
- Total cost: $O(\log(N))$
Heap-Sort: Everything Falls Into Place

• Start with empty heap
• Build the heap: insert $a[0] \ldots a[N-1]$
• Build the result: delete root until heap is empty, gets keys sorted in reverse order
• Use $a$ to store both the array and the heap (explained in lecture)
Heap-Sort: Slightly Faster

- Build the heap faster: Max-Heapify
- Explained in lecture
- $O(N)$ instead of $O(N \cdot \log(N))$
- Total time for Heap-Sort stays $O(N \cdot \log(N))$ because of $N$ deletions
- Max-Heapify is very useful later
Priority Queues

• Data Structure
  • `insert(key)` : adds to the queue
  • `max()` : returns the maximum key
  • `delete-max()` : deletes the max key
  • `delete(key)` : deletes the given key
  • optional (only needed in some apps)
Priority Queues with Max-Heaps

• Doh? (assuming you paid attention so far)
• Costs (see above line for explanations)
  • insert: $O(\log(N))$
  • max: $O(1)$
  • delete-max: $O(\log(N))$
  • delete: $O(\log(N))$ - only if given the index of the node containing the key
Cool / Smart Problem

• Given an array $a$ of numbers, extract the $k$ largest numbers

• Want good running time for any $k$
Cool / Smart Problem

- Small cases:
  - $k = 1$: scan through the array, find $N$
  - $k$ small
    - try to scale the scan
    - getting to $O(kN)$, not good
Cool / Smart Problem

- **Solution:** Heaps!
  - build heap with Max-Heapify
  - delete root \( k \) times
  - \( \mathcal{O}(k \cdot \log(N)) \)

- **Bonus Solution:** Selection Trees (we’ll come back to this if we have time)
Discussion: Priority Queue Algorithms

- BSTs
  - store keys in a BST

- Regular Arrays
  - store keys in an array

- Arrays of Buckets
  - $a[k]$ stores a list of keys with value $k$
And we’re done!

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v. Next

- Get sleep (unmitigated disaster)
- Definitely can’t teach both heaps and sorting
  - Convert heaps to a problem, since they should know the basics from lecture
- Make sorting shorter