Quiz 2

- Do not open this quiz booklet until directed to do so. Read all the instructions on this page.
- When the quiz begins, write your name on every page of this quiz booklet.
- You have 120 minutes to earn 120 points. Do not spend too much time on any one problem. Read them all through first, and attack them in the order that allows you to make the most progress.
- This quiz booklet contains 8 pages, including this one. Two extra sheets of scratch paper are attached. Please detach them before turning in your quiz at the end of the exam period.
- This quiz is closed book. You may use **two** $8\frac{1}{2}'' \times 11''$ or A4 crib sheet (both sides). No calculators or programmable devices are permitted. No cell phones or other communications devices are permitted.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Pages may be separated for grading.
- Do not waste time and paper rederiving facts that we have studied. It is sufficient to cite known results.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.
- Good luck!

Problem	Parts	Points	Grade	Grader
1	20	4		
2	32	4		
3	24	1		
4	24	1		
5	20	3		
Total		120		

Name: ______________________________

Friday Recitation:	Rishabh	Rob	Chieu	Jason	Matthew
	10 AM	**12 PM**	**1 PM**	**2 PM**	**3 PM**

Problem 1. True or False [20 points] (4 parts)

For each of the following questions, circle either T (True) or F (False). **Explain your choice.** (No credit if no explanation given.)

(a) T F Topological sort requires $\Omega(V \lg V)$ if the edge weights are unbounded.
Explain:

(b) T F A set of n integers whose values are in the range $[0, n^8)$ can be sorted in $O(n)$ time.
Explain:

(c) T F If a depth-first analysis of a graph contains at least one back edge, any other depth-first analysis of the same graph will also contain at least one back edge.
Explain:

(d) T F Any DFS forest of an undirected graph contains the same number of trees.
Explain:

Problem 2. Short Answer [32 points] (4 parts)

(a) In the graph of 2x2x2 Rubik's cube positions (as in Problem Set 4), there are exactly 6 edges incident on each vertex. We want to use a bi-directional BFS to find a shortest path between two vertices s and t. If the shortest path from s to t contains d edges, approximately how many vertices must be explored to find a shortest path, in the worst case? How does this compare to the number of vertices required for a single BFS starting from s?

(b) Each edge in a connected, unweighted graph G is colored either red or blue. Present an algorithm to compute a path between s and t that traverses the fewest number of red edges. Analyze its running time.

(c) An efficiency-crazed (complexity minded) algorithms developer tries to reduce the amount of work done in the Bellman-Ford algorithm. Her modified Bellman-Ford first runs a modified BFS starting from s that keeps track of the BFS step $d(e)$ in which it discovers each edge e. Thus, $1 \leq d(e) < |V|$. In the Bellman-Ford algorithm itself, instead of relaxing *all* the edges in every iteration, at iteration i (i starts at 1 and goes up to $|V| - 1$), she only relaxes the edges with $d(e) \leq i$. Is this be guaranteed to give the shortest distance from the s to all the vertices or detect a negative cycle? Explain why or provide a small counterexample.

(d) Describe a strategy for analyzing a directed acyclic graph using depth first search, choosing the search order so that *all* the edges in the analysis will be cross edges. (In other words, the search will produce *no* tree edges, back edges, or forward edges.)

Problem 3. A Vacation Home in Hawaii [24 points]

Beverly owns a vacation home in Hawaii and wishes to rent the place out for n days beginning on May 1st (on the $n + 1$st day, she plans to take a vacation there herself). She has obtained m bids, each of which has a starting day s_i and ending day e_i (between 1 and n), and the amount $\$_i$ that the bidder is willing to pay. Beverly can only rent the house to a single bidder on any given day. (That is, she may not accept two bids b_i and b_j such that the intervals (s_i, e_i) and (s_j, e_j) overlap.)

Beverly decides to model this problem as a directed graph with weighted edges so that she can use a standard graph algorithm (or a minor variation of a standard algorithm) to find the bids to accept which maximizes her revenue.

Describe such a model, and give the asymptotic cost of finding the set of bids that maximizes the revenues. Assume that the bids are given as list of tuples, not necessarily sorted in any particular order. Try to find an algorithm that is as efficient as possible.

Problem 4. Clusters [24 points]

At Podunk State University, there are 100 courses, and n students. The number of students n varies from term to term and is virtually unbounded because Podunk State is required by law to admit all students who apply for admission (and even a few who don't).

Since there are few courses and many students, it is common to find groups of students who are all taking the same set of courses at the same time. In order to optimize its available resources in light of the current economic crisis, the administration wants to know how many clusters there are, where a *cluster* is defined as a maximal set of students who are all registered for the same set of courses.

The administration hires you to solve this problem. You are given a table that lists all n students and the courses each is taking. Design an efficient algorithm to determine the number of clusters. Analyze its running time as a function of n.

Problem 5. Cap-tain Jean-Luc Pi-card, U-S-S En-ter-prise [20 points] (3 parts)

The Enterprise is in orbit around Starbase 6006. An urgent medical situation requires that it travel as quickly as possible to the distant planet *Vertex T*. In fact, the situation is so urgent that the Federation requests that the Enterprise arrive *yesterday*. Your goal is to determine if it is possible to get the Enterprise from s to t in negative total time, so that it arrives before it leaves.

You have a directed graph G representing the Universe. Each known location in Federation space is a vertex in the graph. There are a number of hyperspace bypasses between these points; a hyperspace bypass from point u to point v is represented by the edge (u, v). The weight of the edge $w(u, v)$ is equal to the amount of time that passes while traversing it.

Notice that edges may have negative weights! Some bypasses run through some sort of crazy time-warp thing, causing the ship to travel back in time.

(a) [2 points] Ensign Dijkstra volunteers to lay in the course. In one sentence, explain why Dijkstra's algorithm can't be used to solve this problem.

(b) [2 points] You ask Ensign Bitdiddle to plot the course instead. Supposing that G contains no negative-weight cycles, describe an algorithm that returns TRUE if there exists a path from s to t with negative total weight, and FALSE otherwise.

Analyze the running time of your algorithm.

(c) [16 points] Lieutenant Data tells you that there is exactly one negative-weight cycle present in G. Describe an algorithm that returns TRUE if there exists a path from s to t with negative total weight, and FALSE otherwise. (You may assume that you know which vertices are in the cycle.)

Analyze the running time of your algorithm.

(d) [0 points] (This part is optional.) Ensign Treaps, in his bright red shirt, suggests you use a treap (from Quiz 1) to solve this problem. Fortunately, he is destined to die on arrival at Vertex T. Describe his glorious death.

SCRATCH PAPER

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