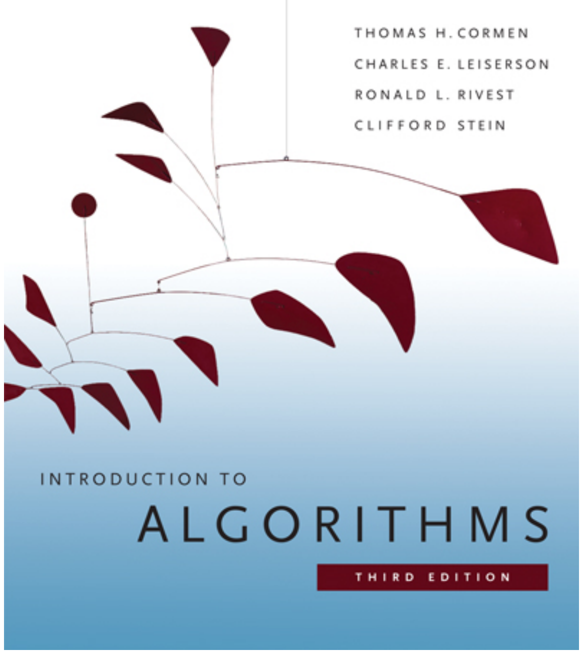
6.006- *Introduction to Algorithms*



Lecture 3

Menu

- Sorting!
  - Insertion Sort
  - Merge Sort
- Solving Recurrences

The problem of sorting

Input: array A[1…n] of numbers.

Output: permutation B[1…n] of A such that $B[1] \le B[2] \le \cdots \le B[n]$.

e.g. $A = [7, 2, 5, 5, 9.6] \rightarrow B = [2, 5, 5, 7, 9.6]$

How can we do it efficiently ?

Why Sorting?

- Obvious applications
  - Organize an MP3 library
  - Maintain a telephone directory
- Problems that become easy once items are in sorted order
  - Find a median, or find closest pairs
  - Binary search, identify statistical outliers
- Non-obvious applications
  - Data compression: sorting finds duplicates
  - Computer graphics: rendering scenes front to back

Insertion sort

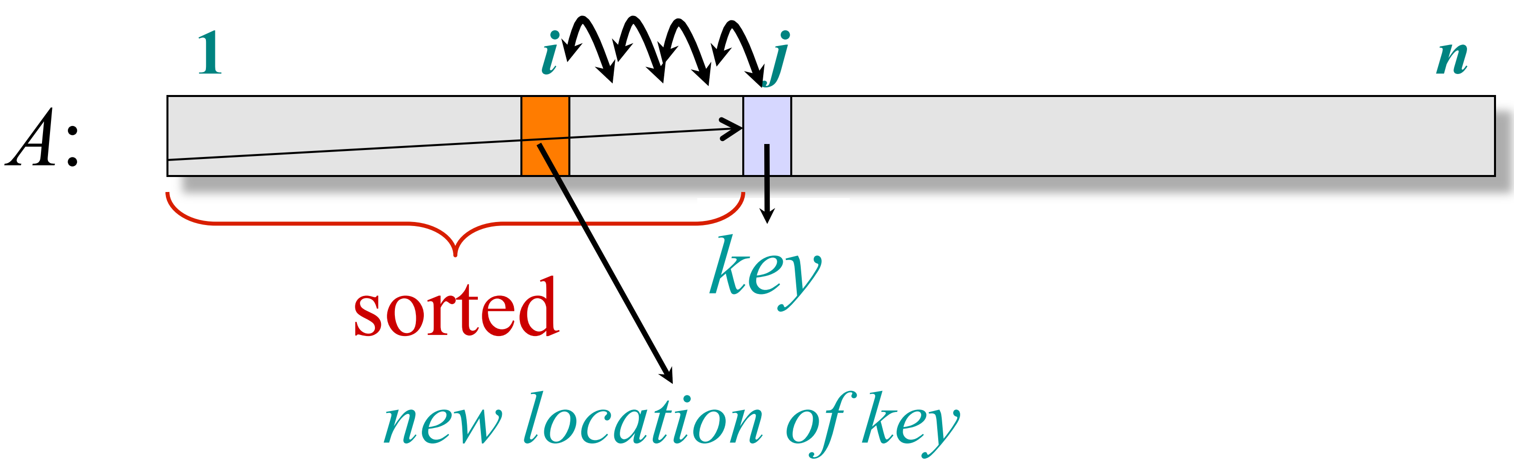
INSERTION-SORT (A, n) $\triangleright$ $A[1 \,.\,.\, n]$

for $j \leftarrow 2$ **to** n

insert key A[j] **into the (already sorted) sub-array** A[1 .. j-1]**.**

by pairwise key-swaps down to its right position

Illustration of iteration *j*



Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

2 4 8 9 3 6

Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

2 4 8 9 3 6

Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6

Example of insertion sort

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Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
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Example of insertion sort

8	2	4	9	3	6	
2	8	4	9	3	6	
2	4	8	9	3	6	
2	4	8	9	3	6	
2	3	4	8	9	6	
2	3	4	6	8	9	*done*

Running time? $\Theta(n^2)$ because $\Theta(n^2)$ compares and $\Theta(n^2)$ swaps

e.g. when input is $A = [n, n-1, n-2, \ldots, 2, 1]$

Binary Insertion sort

BINARY-INSERTION-SORT (A, n) $\triangleright$ $A[1 \,.\,.\, n]$

for $j \leftarrow 2$ **to** n

insert key $A[j]$ **into the (already sorted) sub-array** $A[1 \,..\, j\text{-}1]$**.**

Use binary search to find the right position

Binary search with take $\Theta(\log n)$ time.
However, shifting the elements after insertion will still take $\Theta(n)$ time.

Complexity: $\Theta(n \log n)$ comparisons

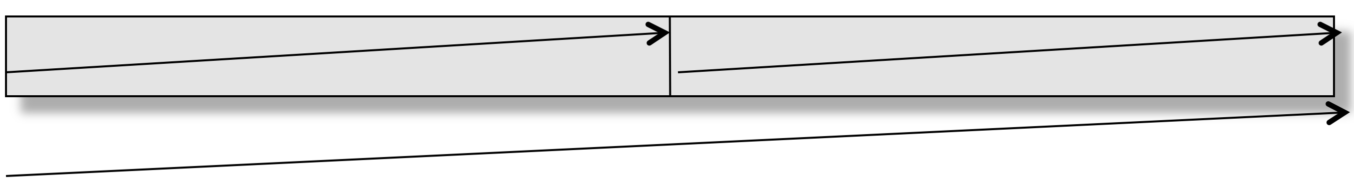
$\Theta(n^2)$ swaps

Meet Merge Sort

divide and conquer

MERGE-SORT $A[1 \mathrel{.\,.} n]$

1. If $n = 1$, done (nothing to sort).
2. Otherwise, recursively sort $A[\,1 \mathrel{.\,.} n/2\,]$ and $A[\,n/2+1 \mathrel{.\,.} n\,]$.
3. "***Merge***" the two sorted sub-arrays.



Key subroutine: **MERGE**

Merging two sorted arrays

20	12
13	11
7	9
2	1

Merging two sorted arrays

20	12
13	11
7	9
2	1

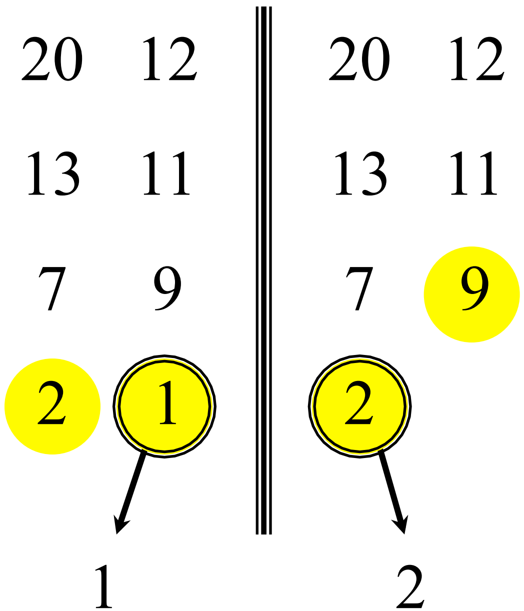
1

Merging two sorted arrays

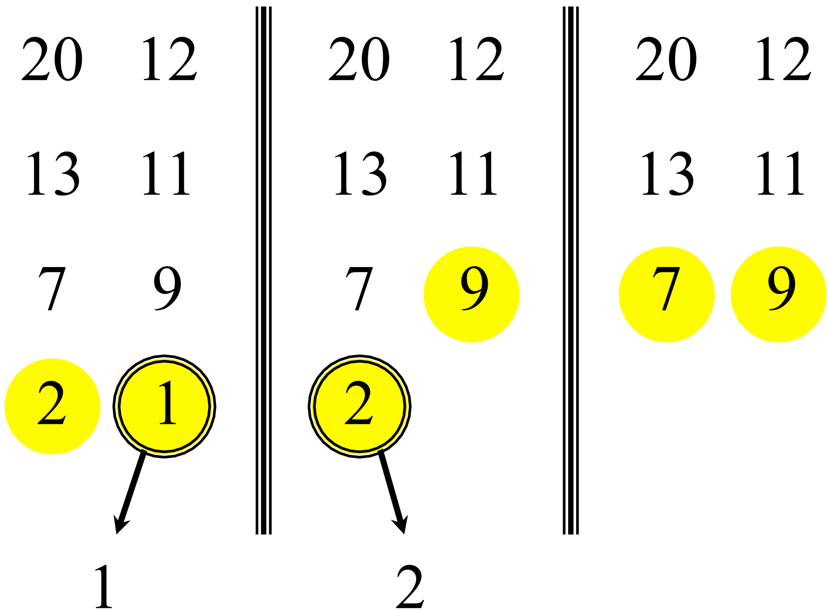
20	12	‖	20	12
13	11	‖	13	11
7	9	‖	7	9
2	1	‖	2	

1

Merging two sorted arrays



Merging two sorted arrays



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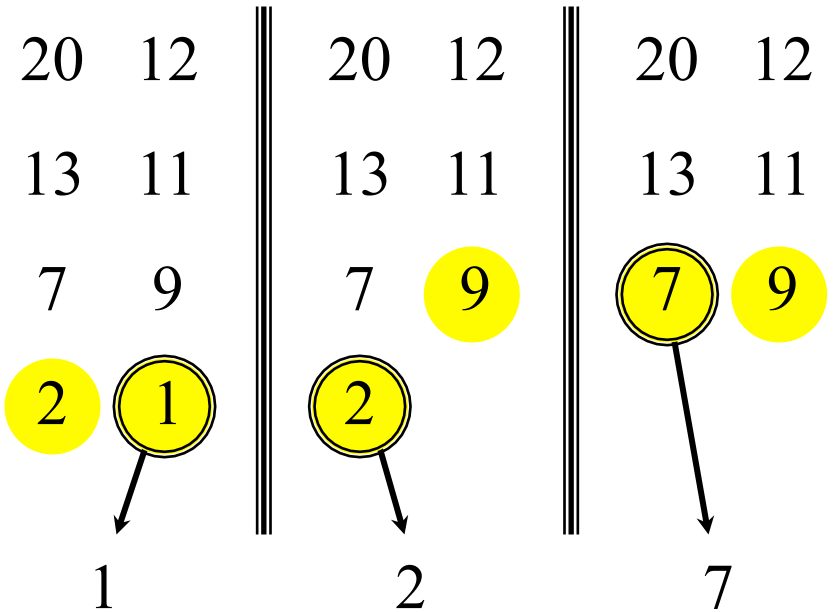

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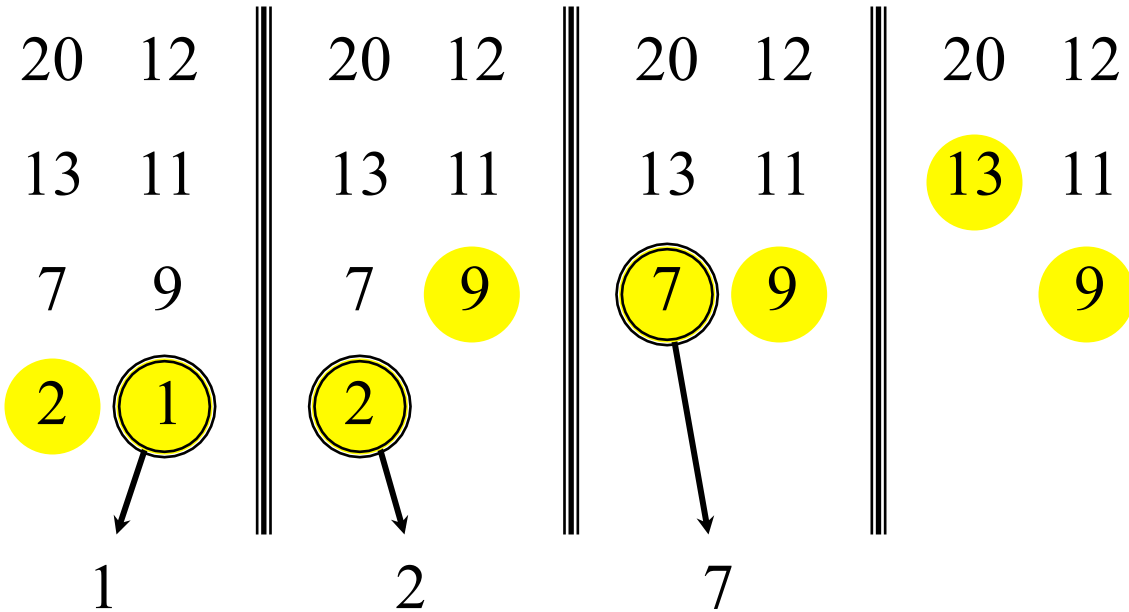

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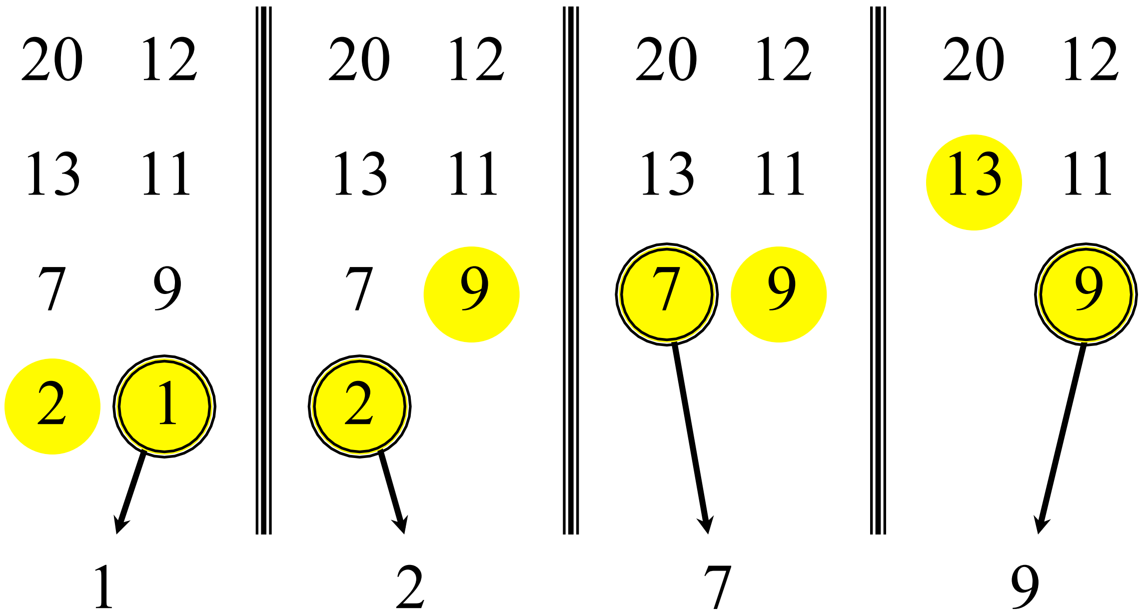

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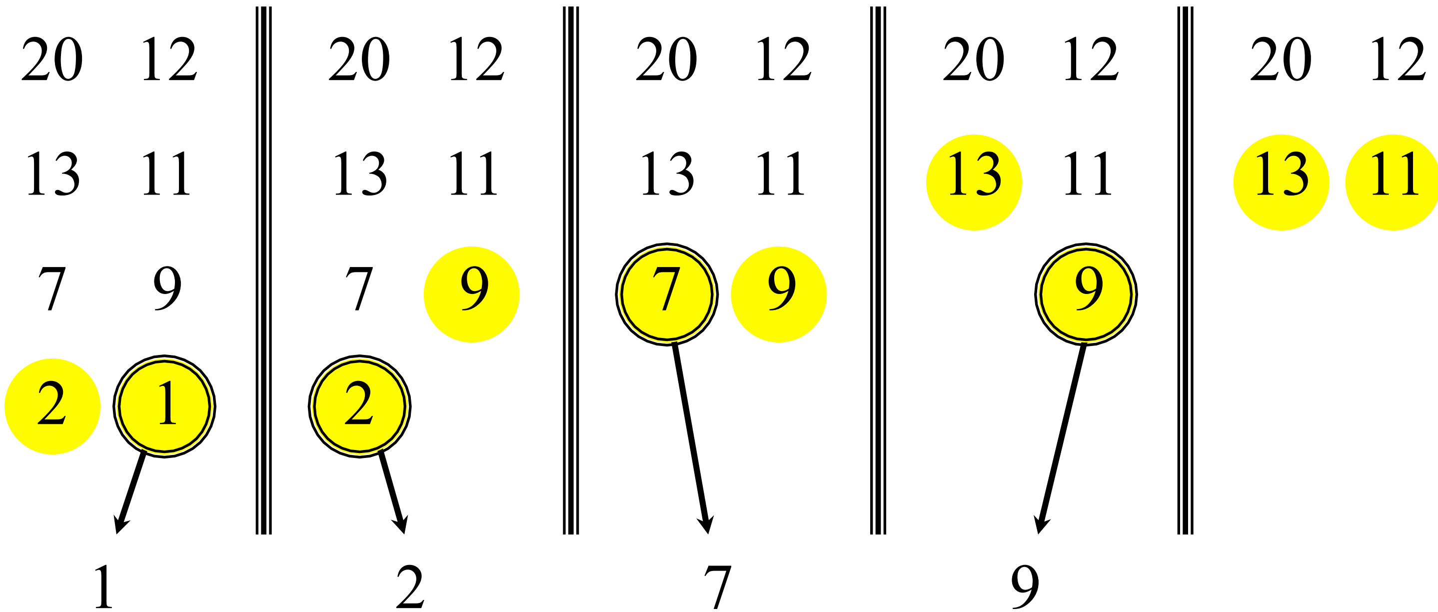

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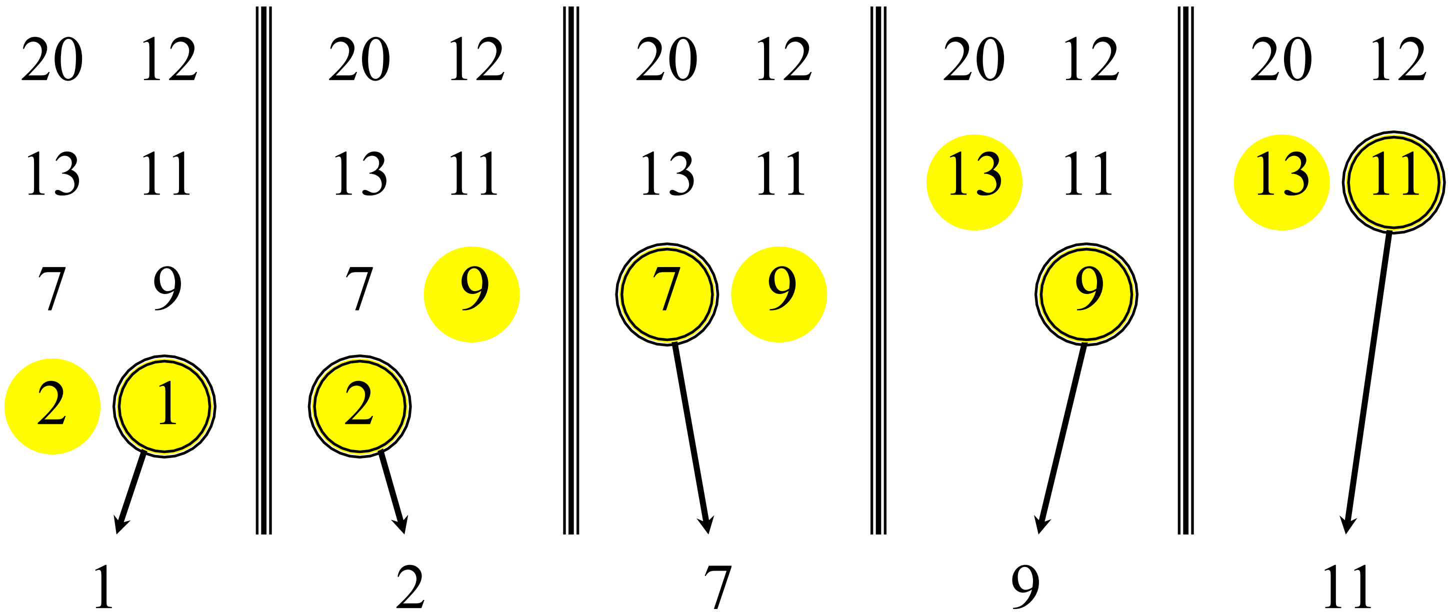
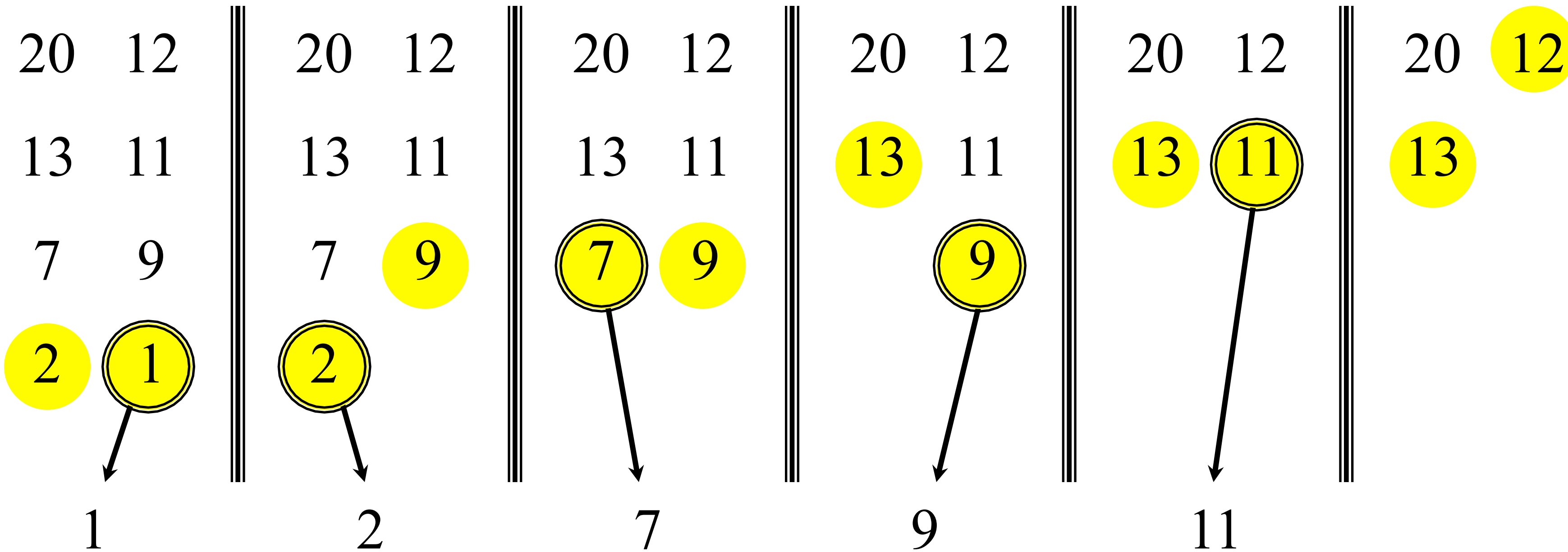
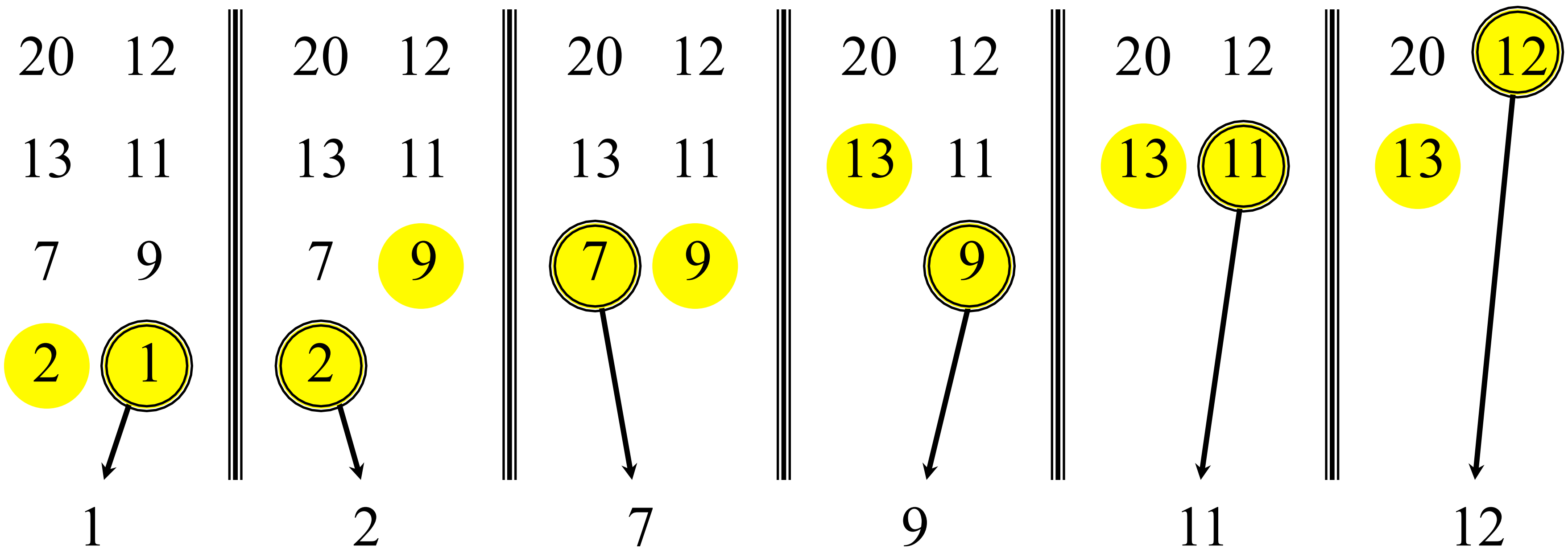


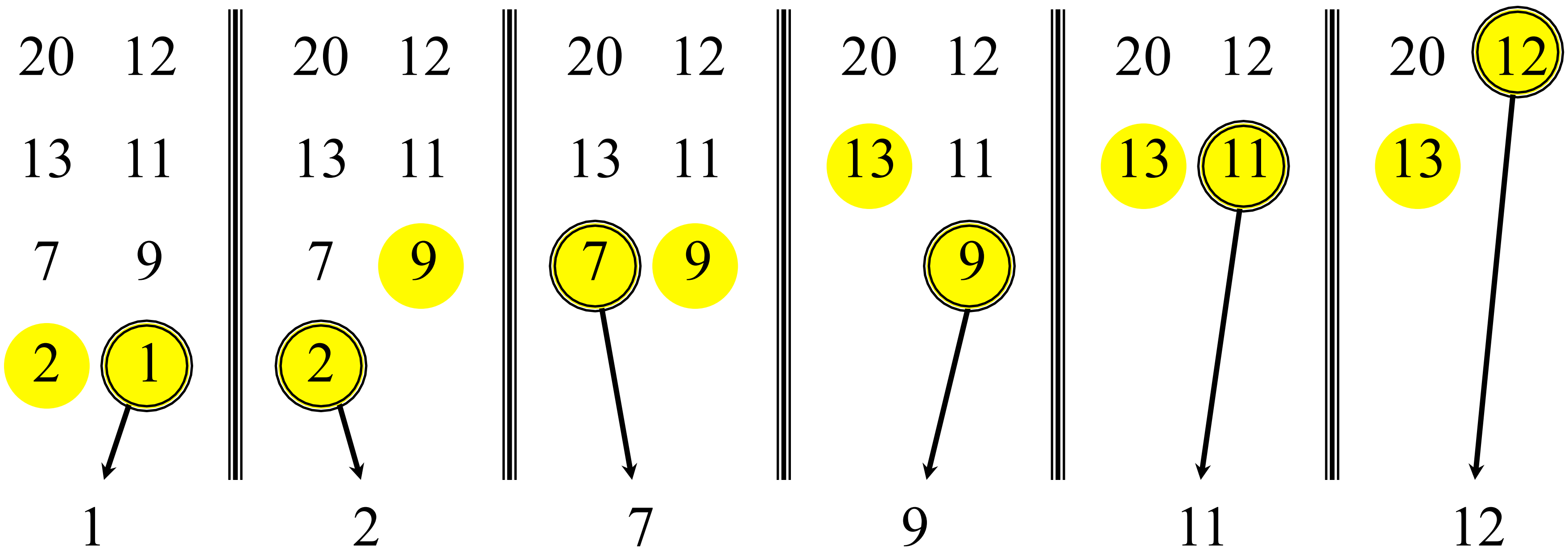
Merging two sorted arrays



Merging two sorted arrays



Merging two sorted arrays



Time = $\Theta(n)$ to merge a total of n elements (linear time).

Analyzing merge sort

MERGE-SORT $A[1 \,..\, n]$	$T(n)$
1. If $n = 1$, done.	$\Theta(1)$
2. Recursively sort $A[\,1 \,..\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,..\, n\,]$.	$2T(n/2)$
3. ***"Merge"*** the two sorted lists	$\Theta(n)$

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

$$T(n) = ?$$

Recurrence solving

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$T(n)$$

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$
\begin{array}{ccc}
 & cn & \\
T(n/2) & & T(n/2)
\end{array}
$$

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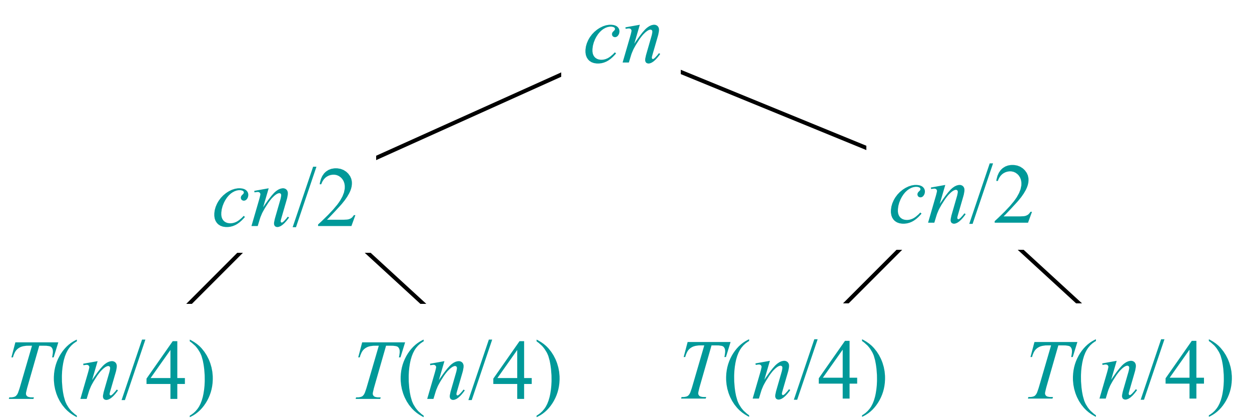

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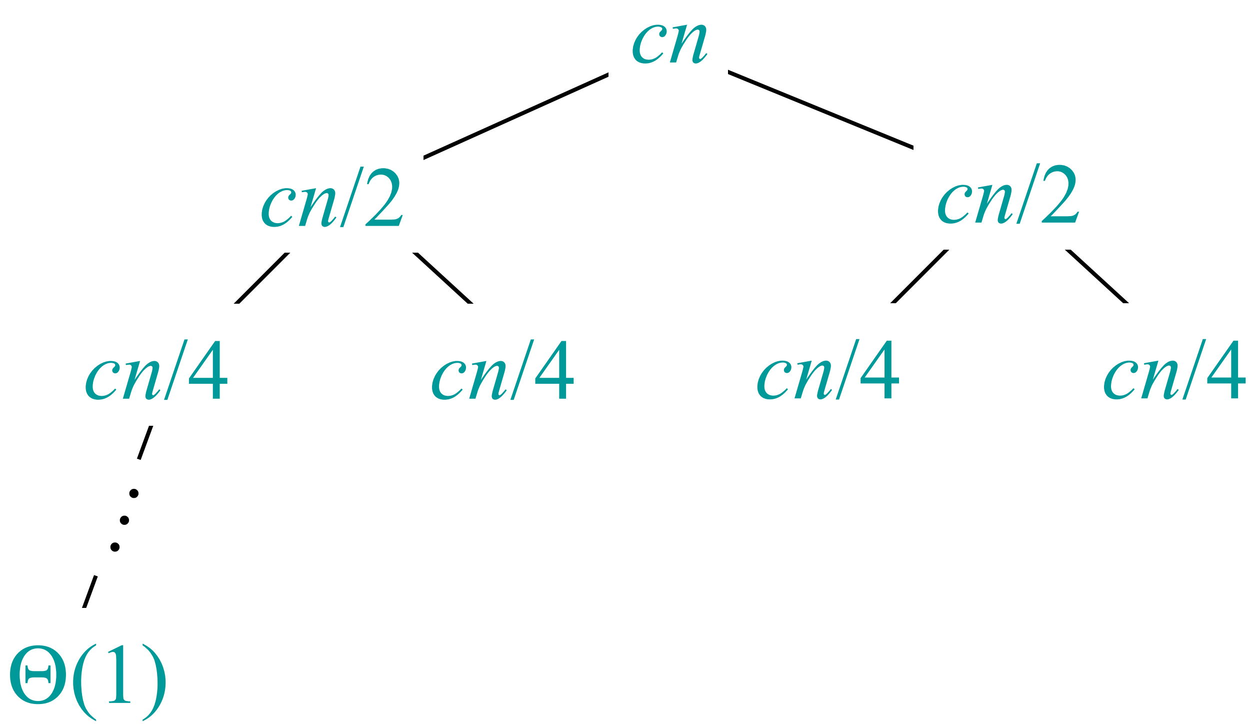

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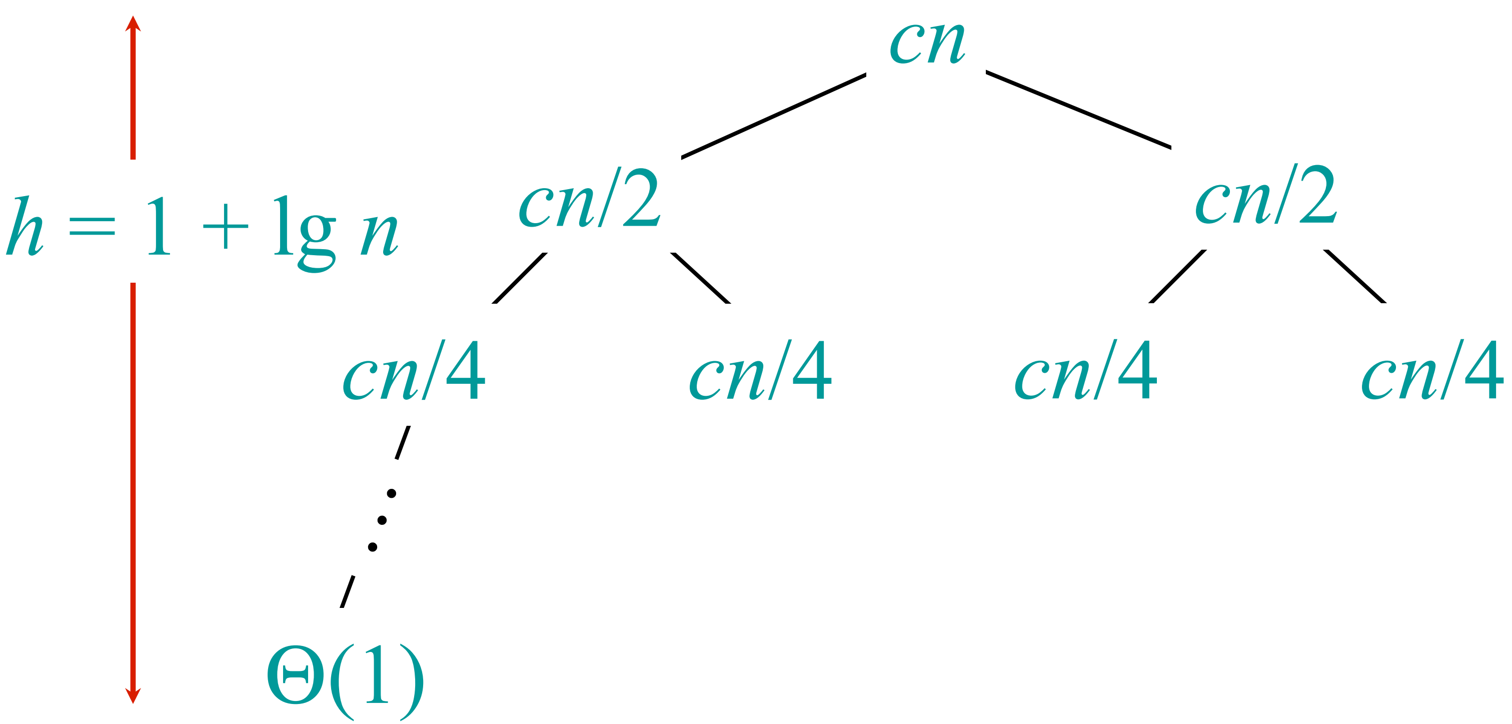

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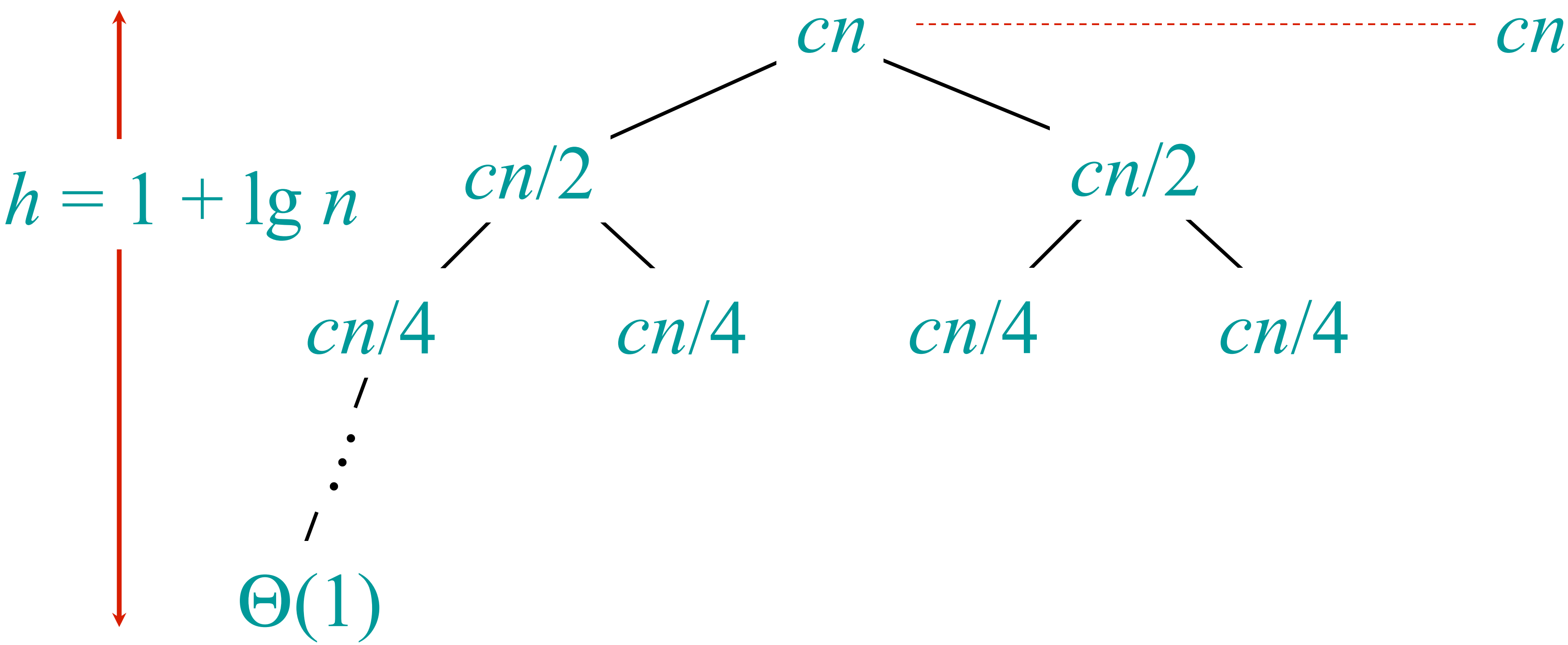

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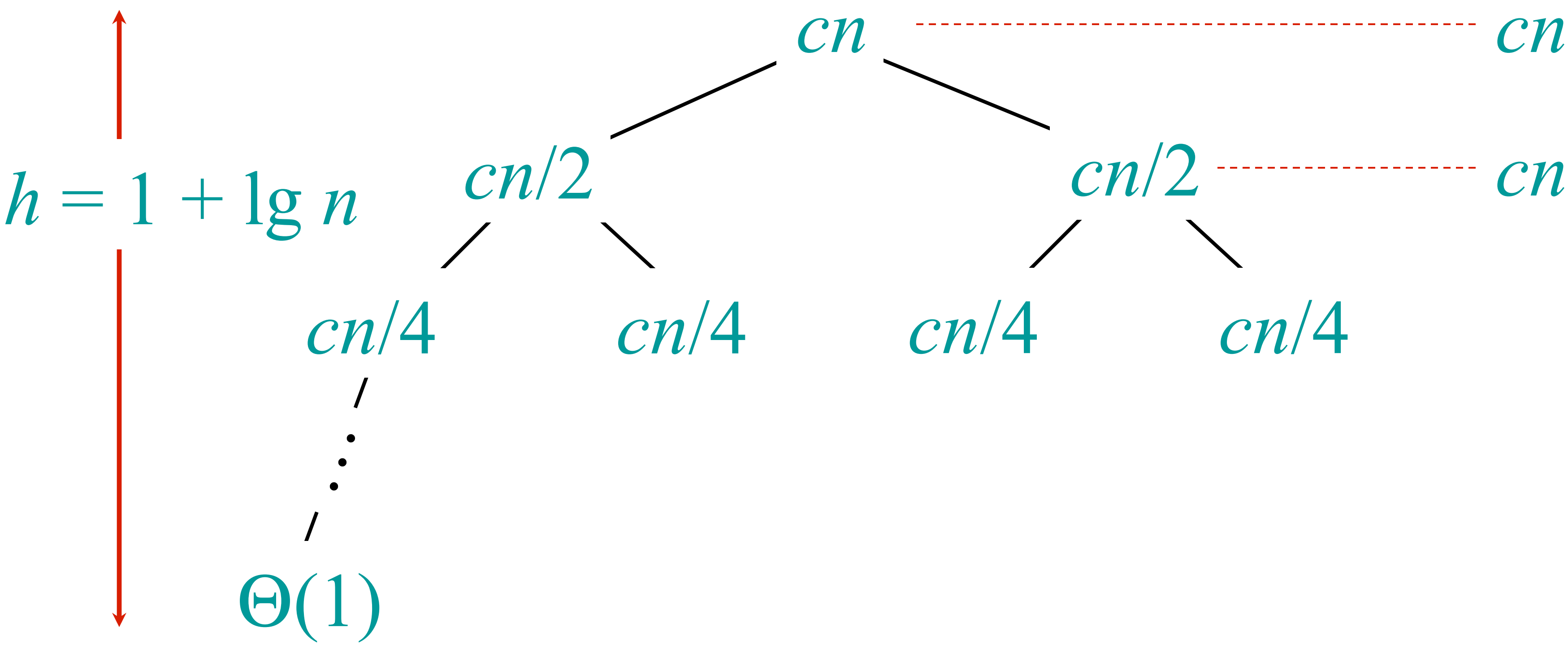

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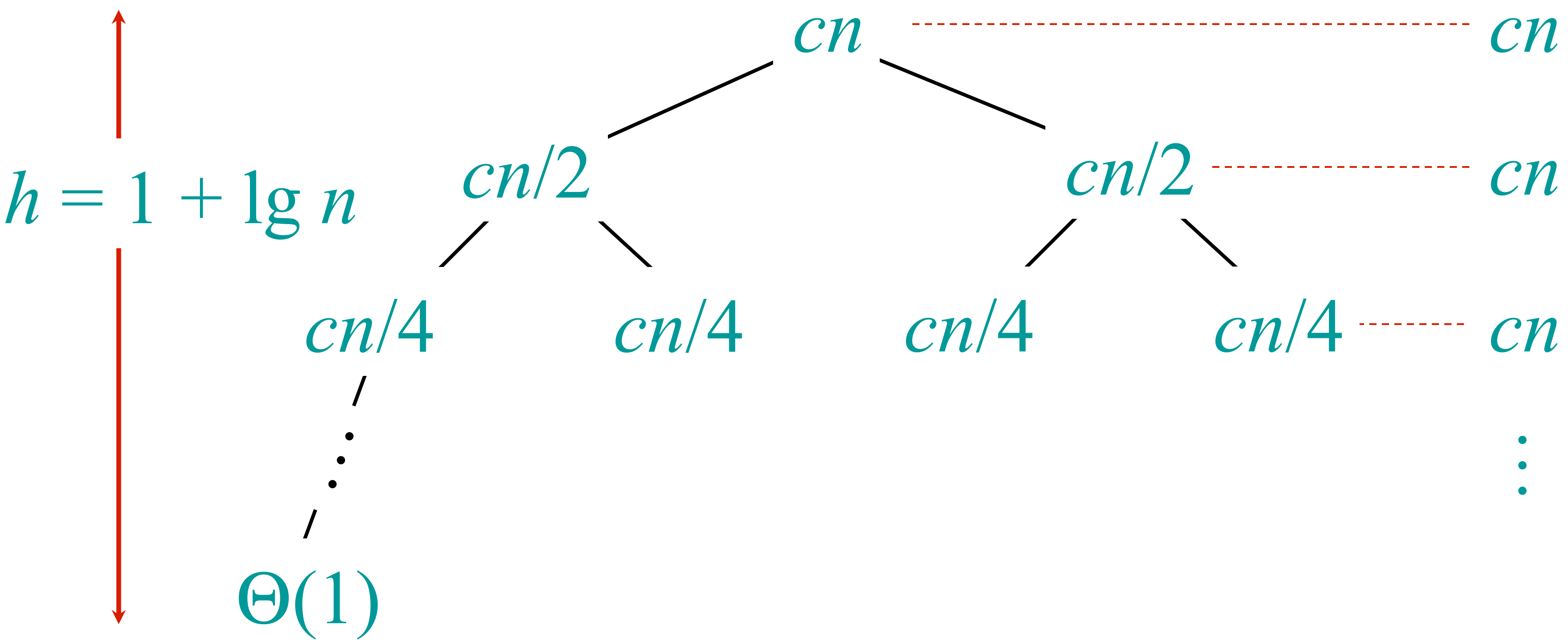

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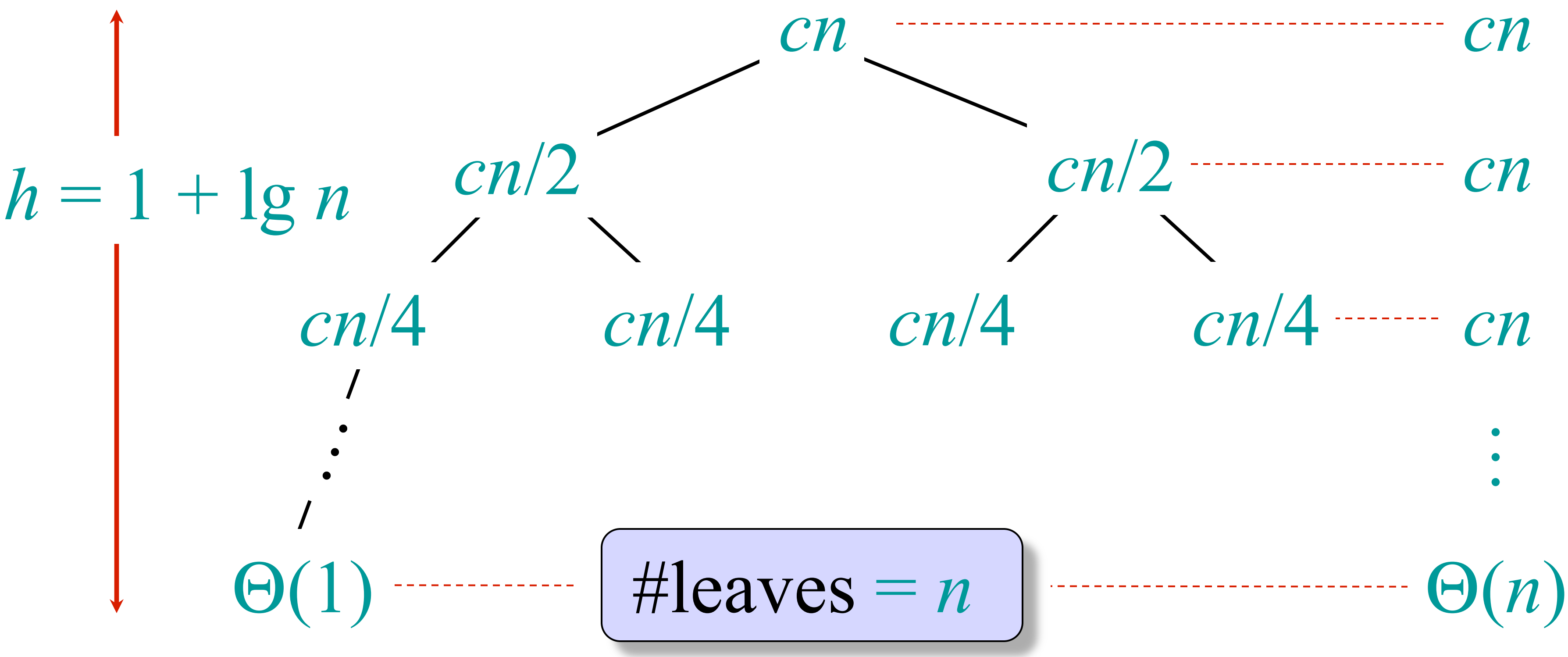

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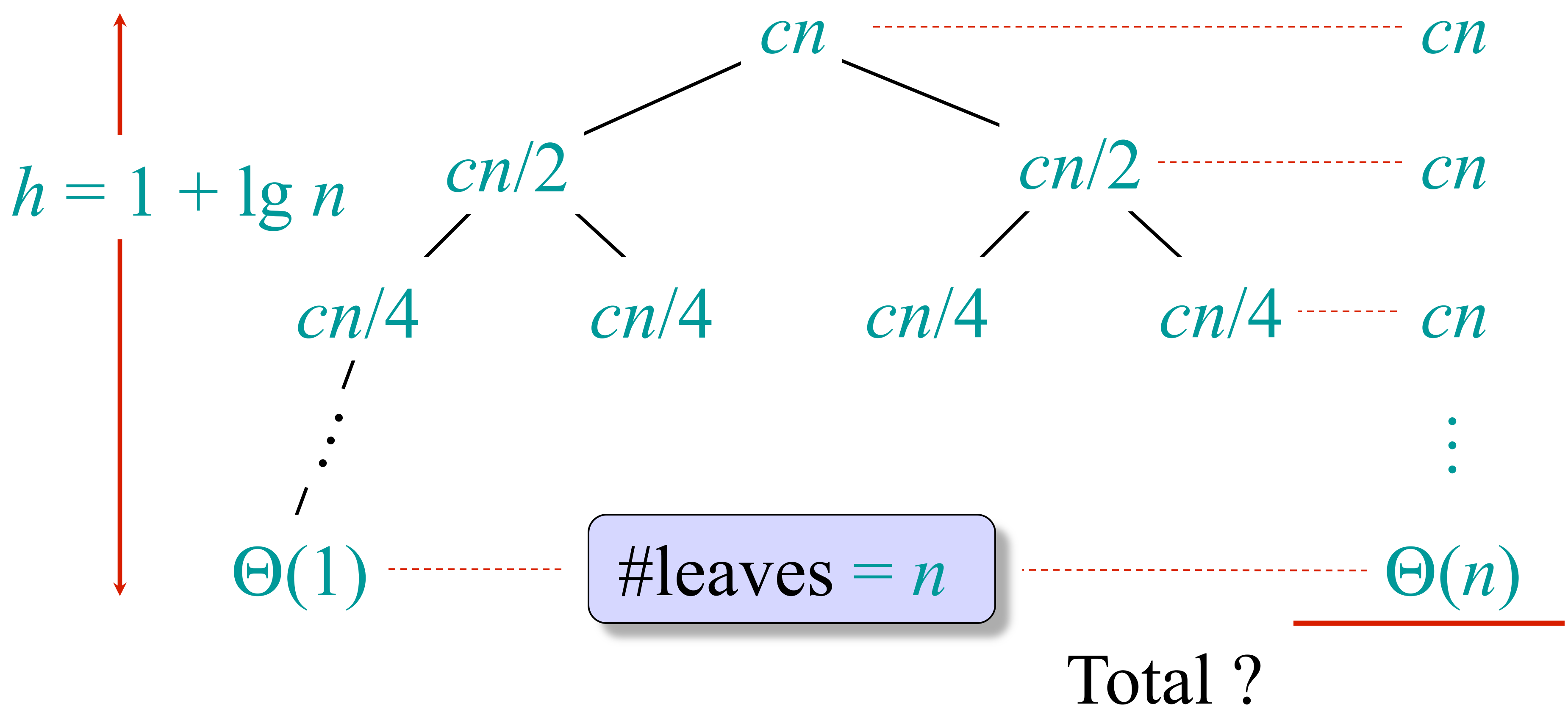

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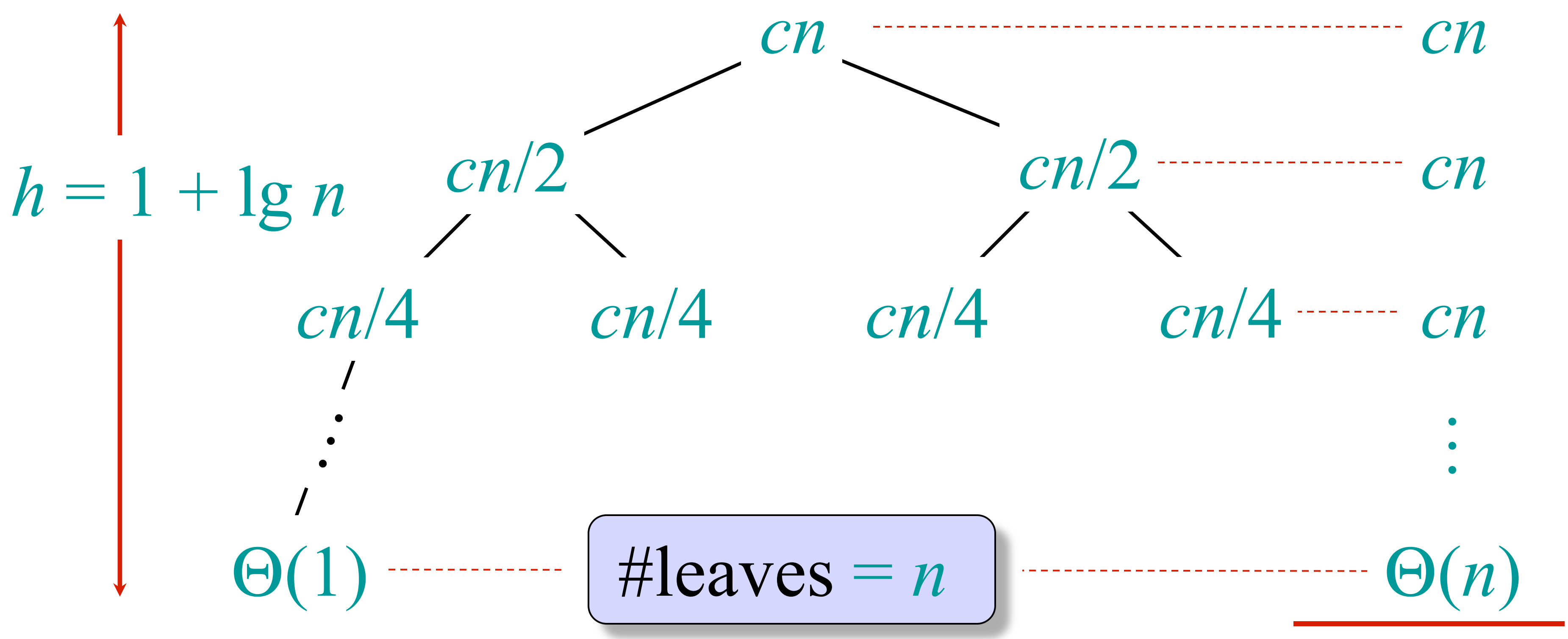
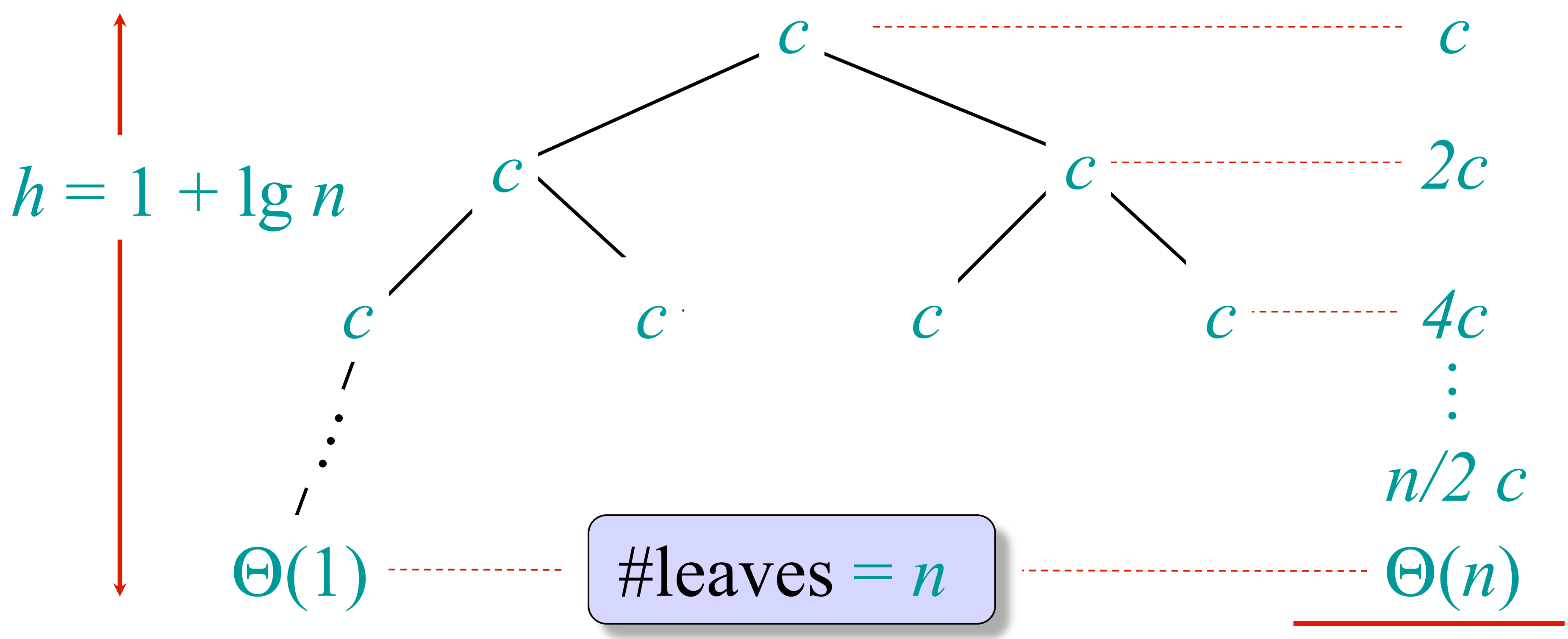


Total $= \Theta(n \lg n)$

Equal amount of work done at each level

Tree for different recurrence

Solve $T(n) = 2T(n/2) + c$, where $c > 0$ is constant.



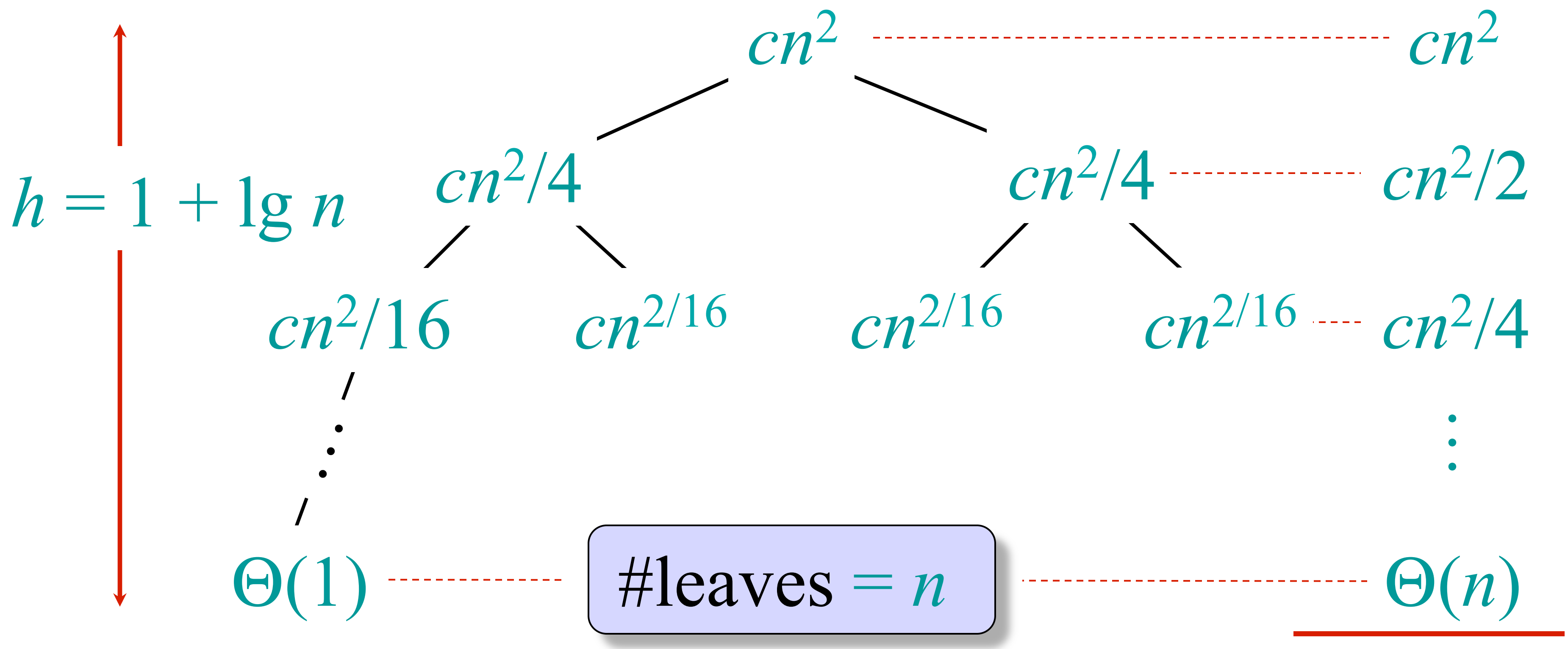
Note that $1 + ½ + ¼ + \ldots < 2$

Total $= \Theta(n)$

All the work done at the leaves

Tree for yet another recurrence

Solve $T(n) = 2T(n/2) + cn^2$, $c > 0$ is constant.



Note that $1 + ½ + ¼ + \ldots < 2$

Total $= \Theta(n^2)$

All the work done at the root