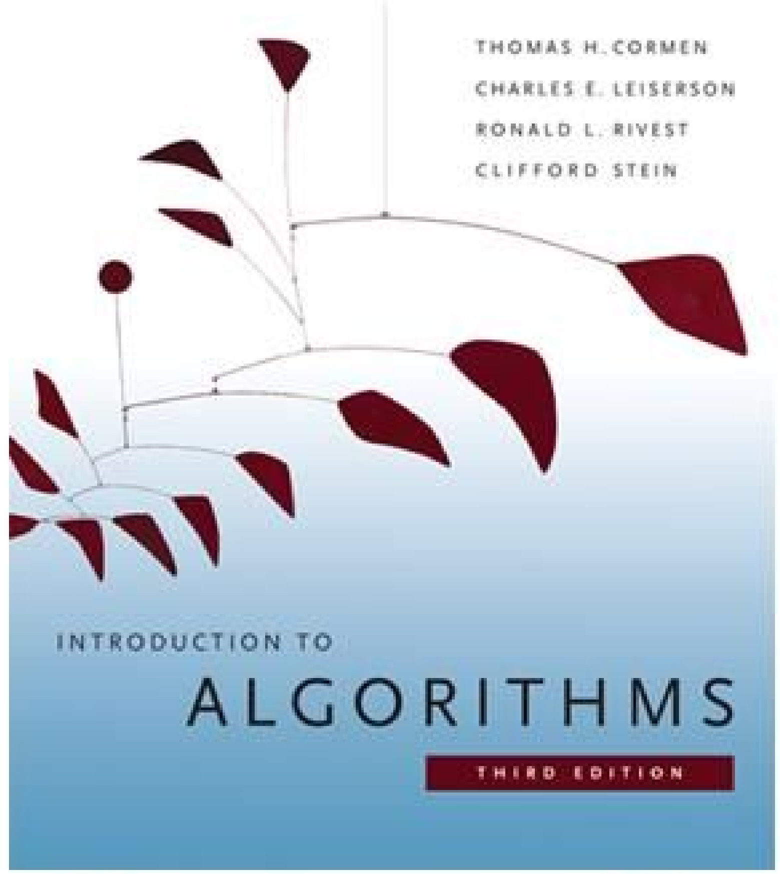
6.006- *Introduction to Algorithms*



Lecture 8

Prof. Constantinos Daskalakis

CLRS: chapter 4.

Menu

- Sorting!
  - Insertion Sort
  - Merge Sort
- Recurrences
  - Master theorem

The problem of sorting

Input: array A[1…n] of numbers.

Output: permutation B[1…n] of A such that $B[1] \le B[2] \le \cdots \le B[n]$.

e.g. A = [7, 2, 5, 5, 9.6] → B = [2, 5, 5, 7, 9.6]

How can we do it efficiently ?

Insertion sort

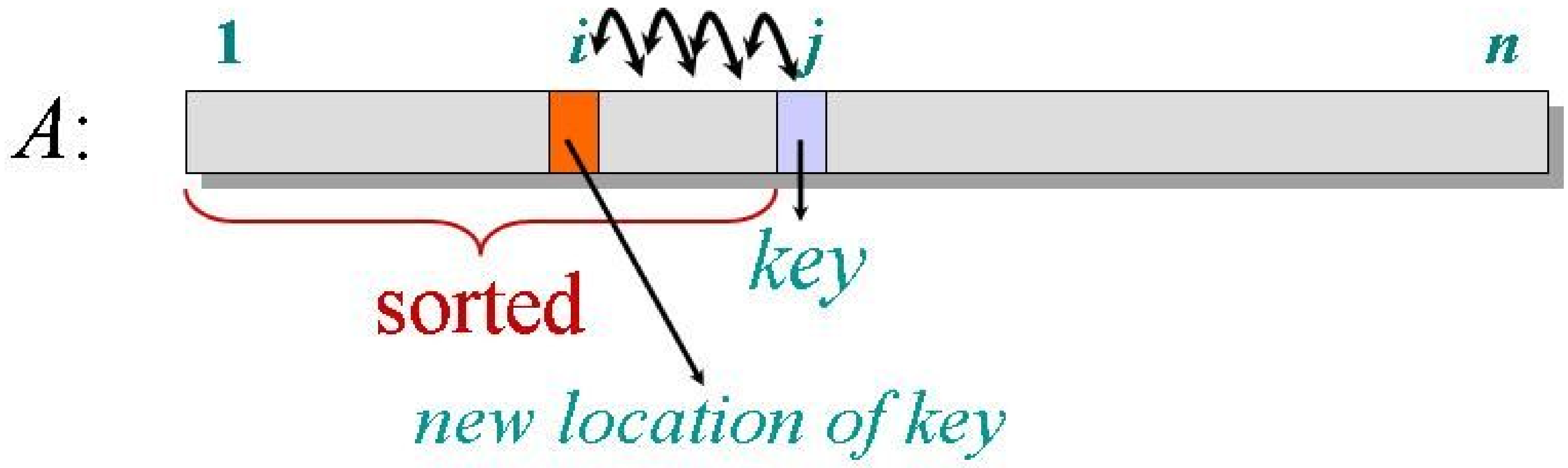
INSERTION-SORT (A, n) $\triangleright$ $A[1 \,..\, n]$

for $j \leftarrow 2$ **to** n

insert key $A[j]$ **into the (already sorted) sub-array** $A[1 \,..\, j\text{-}1]$.

by pairwise key-swaps down to its right position

Illustration of iteration j



Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

8 2 4 9 3 6

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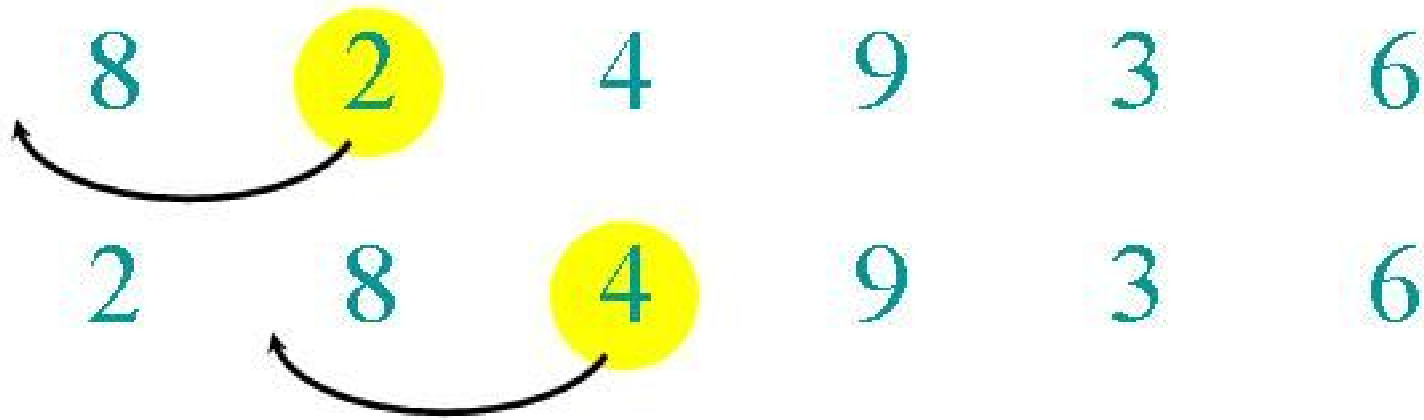
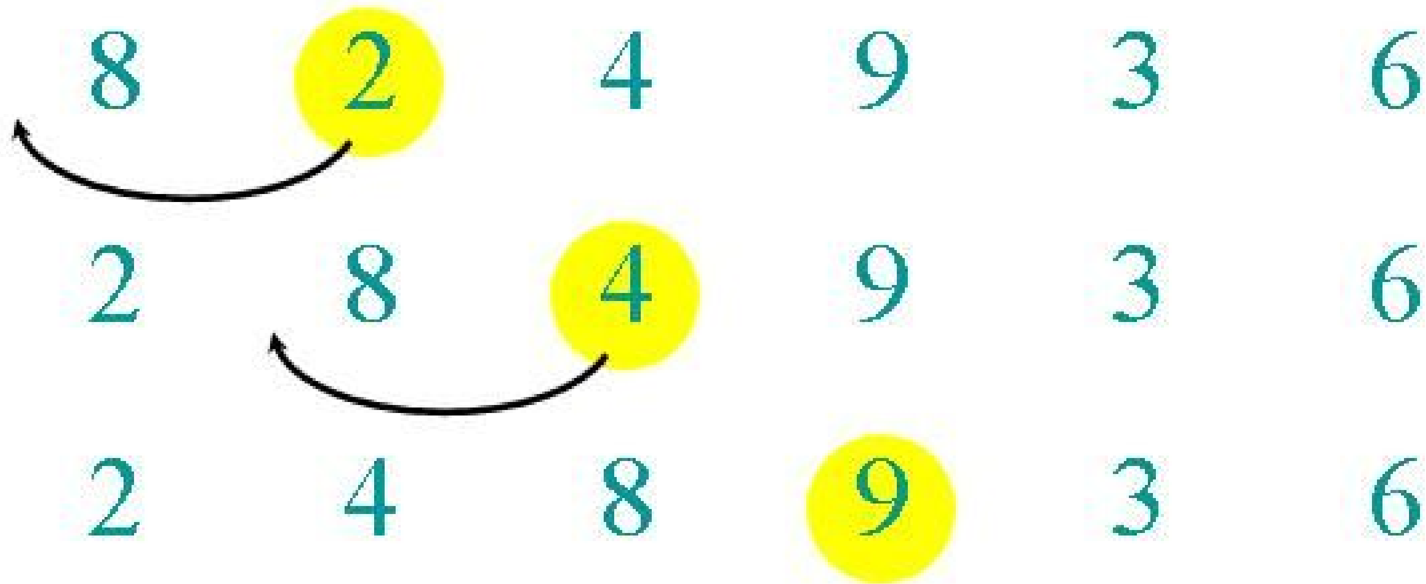
8 2 4 9 3 6

2 8 4 9 3 6

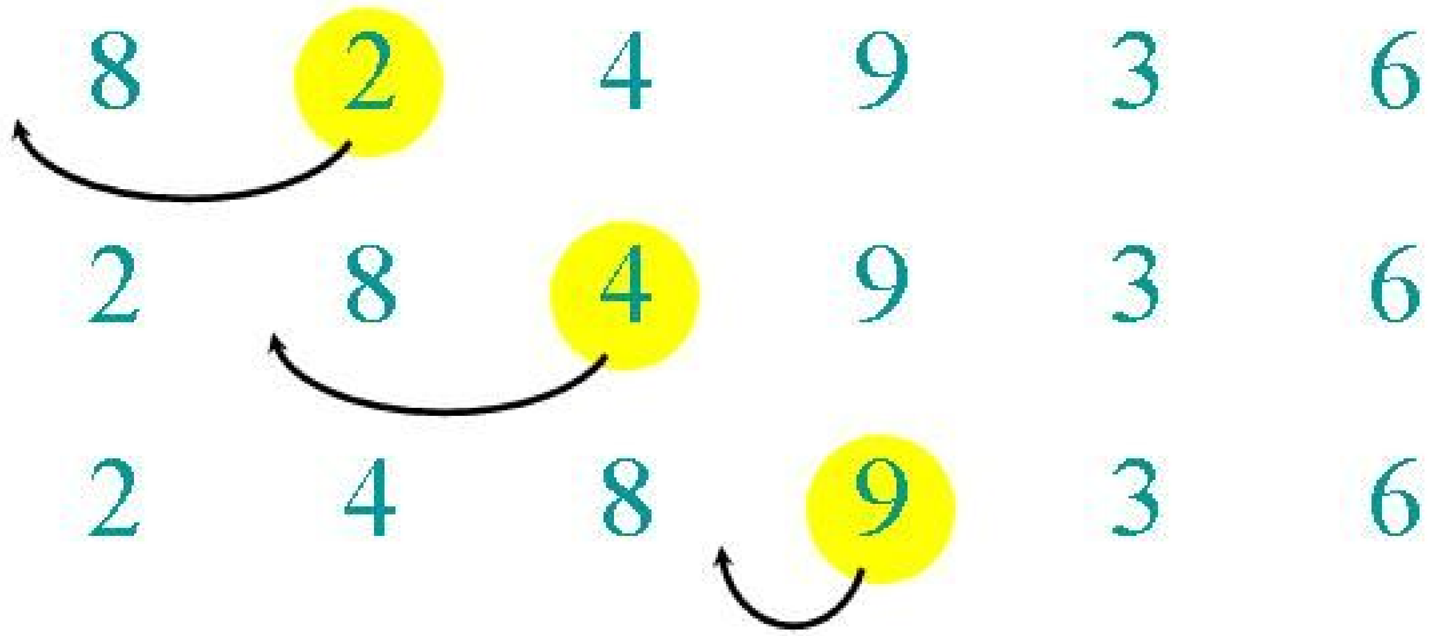
Example of insertion sort



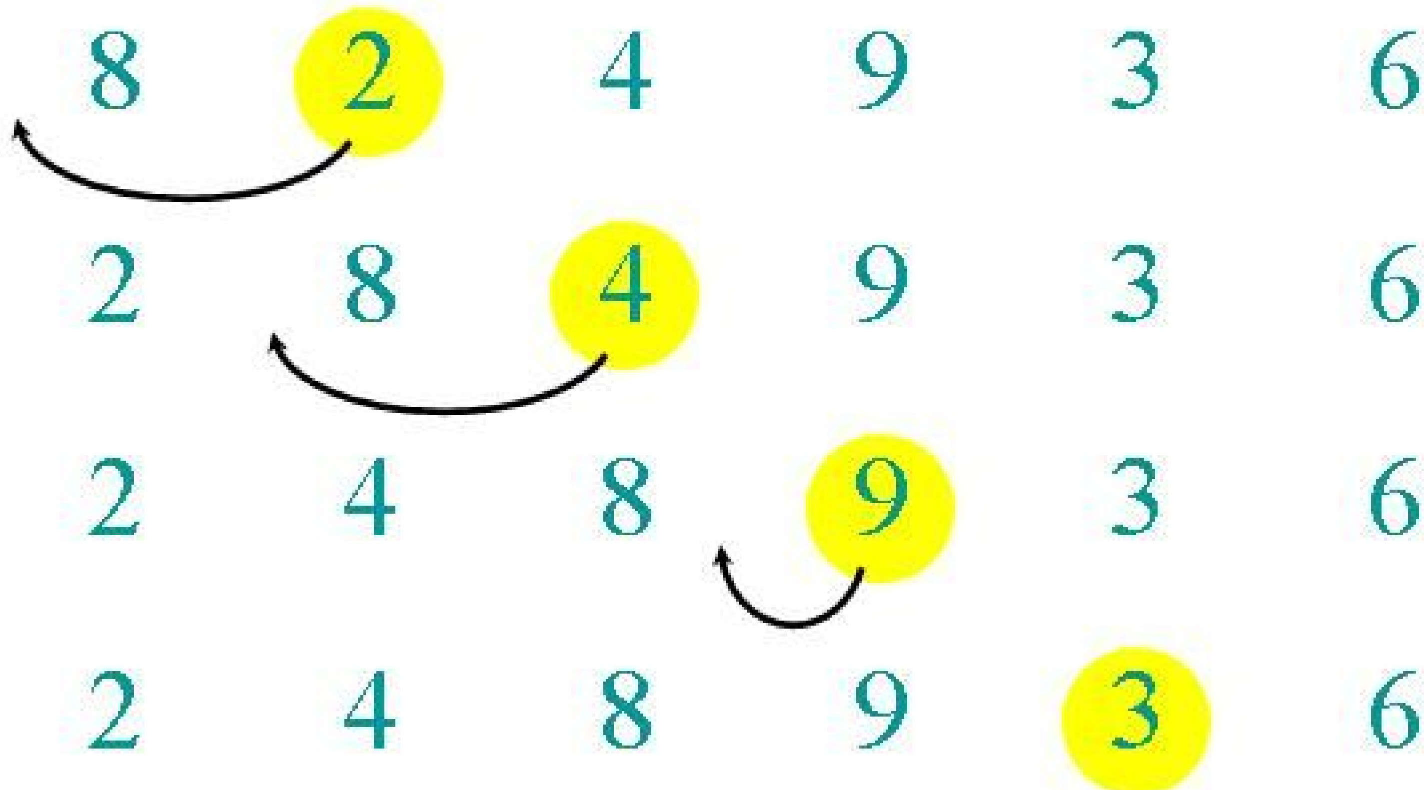
Example of insertion sort



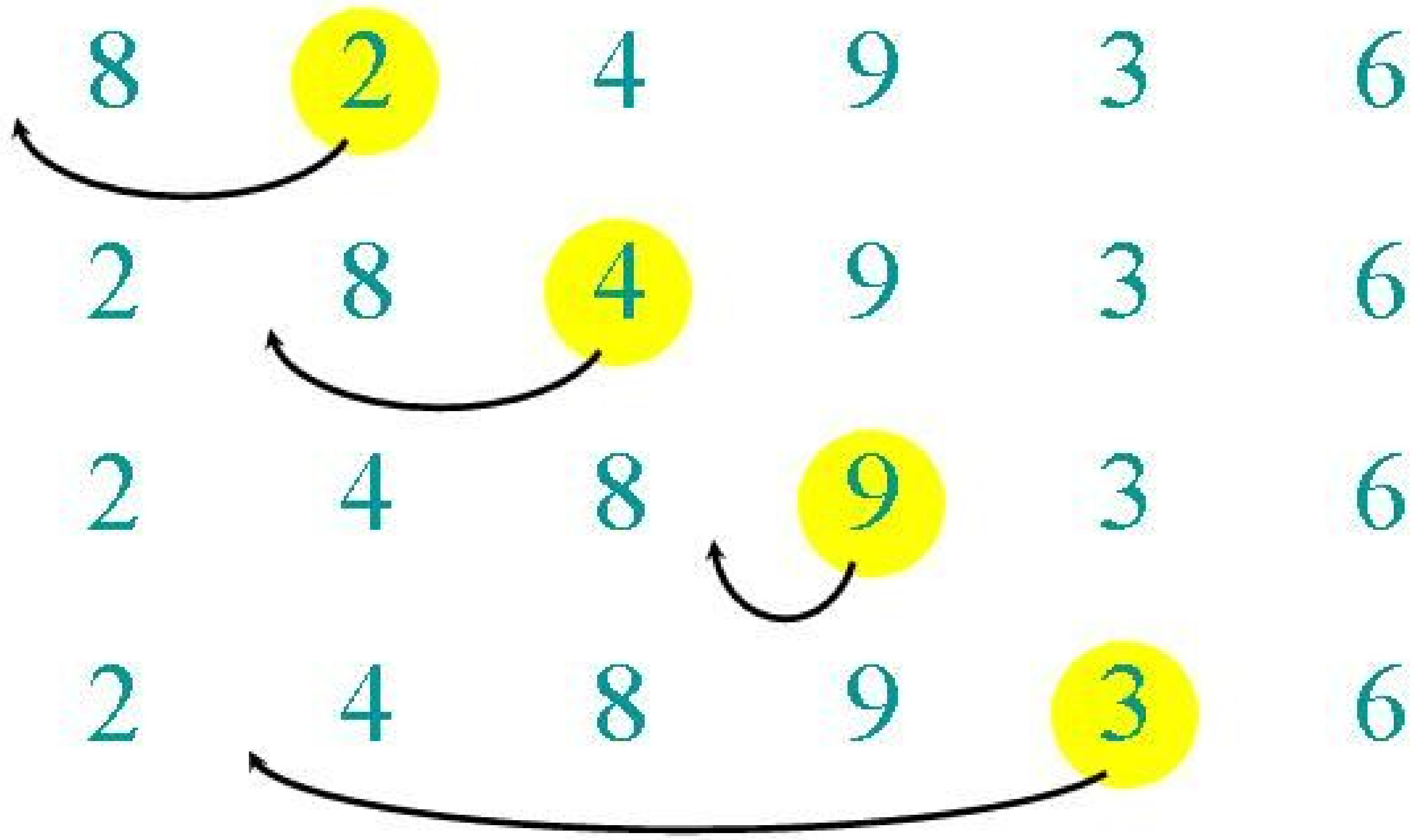
Example of insertion sort



Example of insertion sort



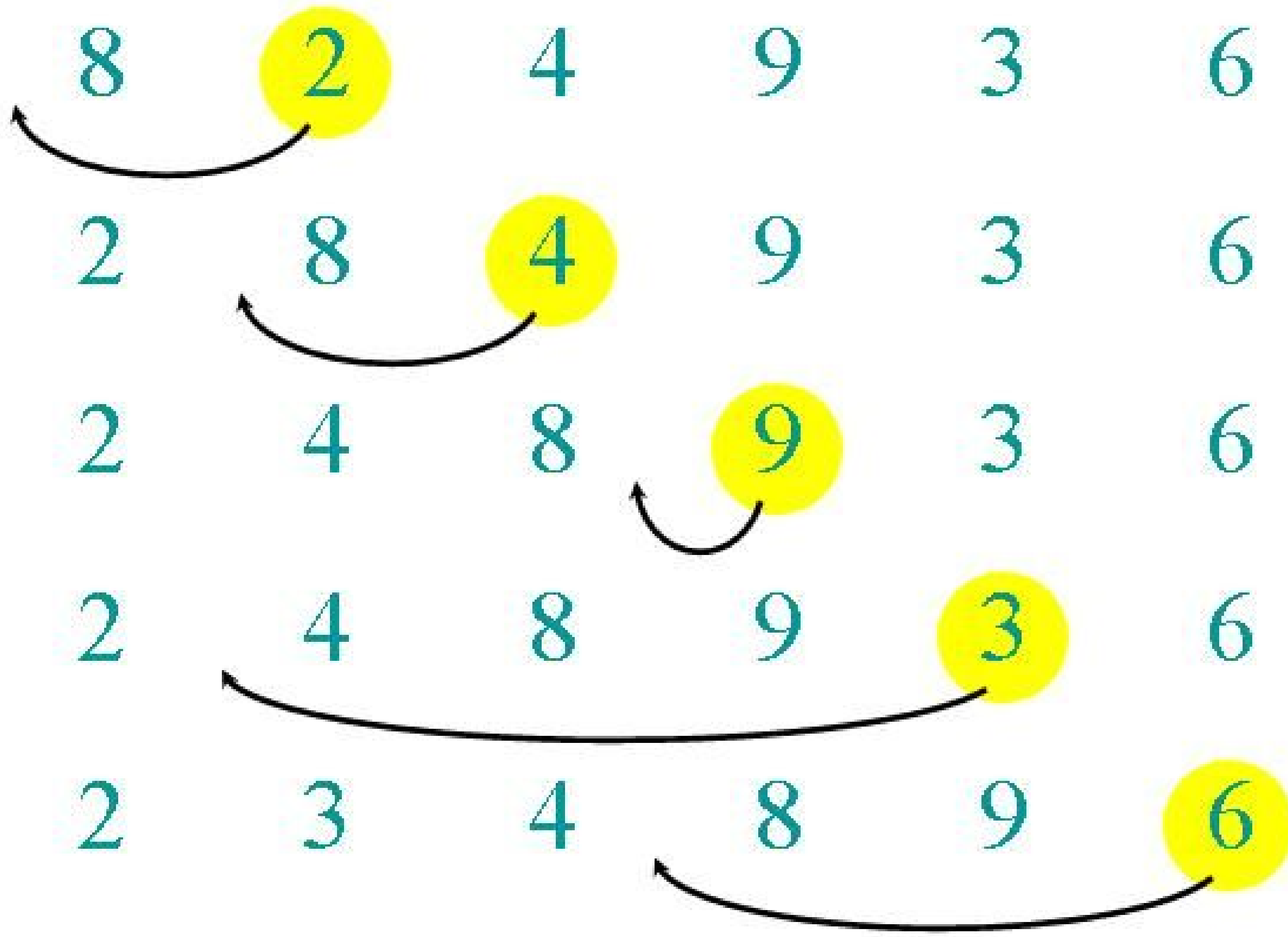
Example of insertion sort



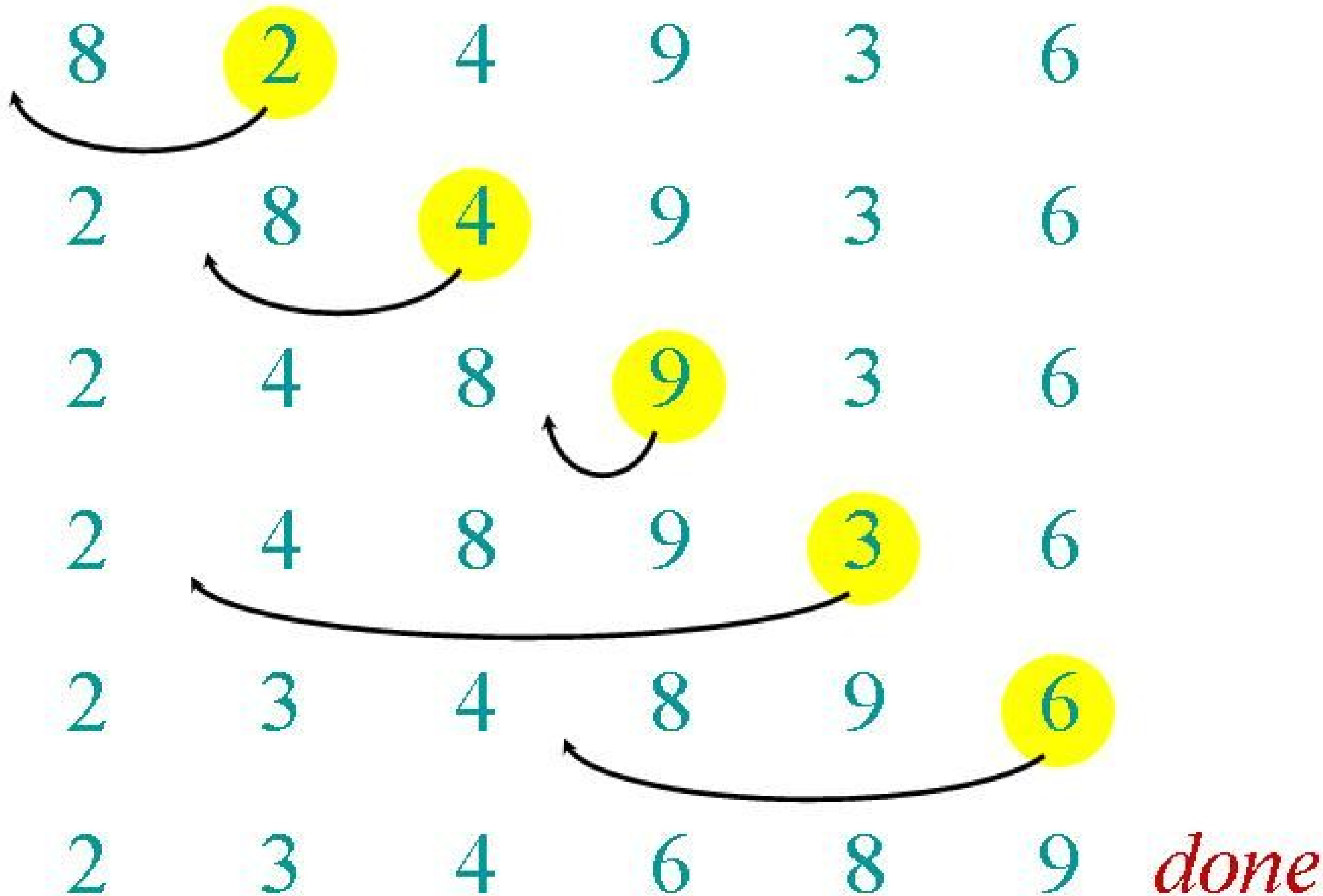
Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6
2	3	4	8	9	6

Example of insertion sort



Example of insertion sort



Running time? $O(n^2)$

e.g. when input is $A = [n, n-1, n-2, \ldots, 2, 1]$

Meet Merge Sort

divide and conquer

MERGE-SORT $A[1 \,.\,.\, n]$

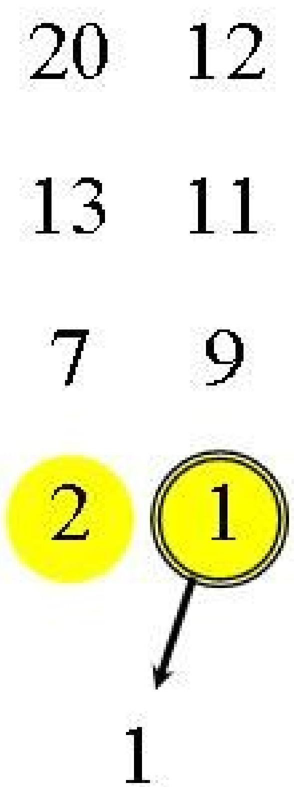
1. If $n = 1$, done (nothing to sort).
2. Otherwise, recursively sort $A[\,1 \,.\,.\, n/2\,]$ and $A[\,n/2+1 \,.\,.\, n\,]$.
3. "***Merge***" the two sorted sub-arrays.

Key subroutine: **MERGE**

Merging two sorted arrays

20	12
13	11
7	9
2	1

Merging two sorted arrays



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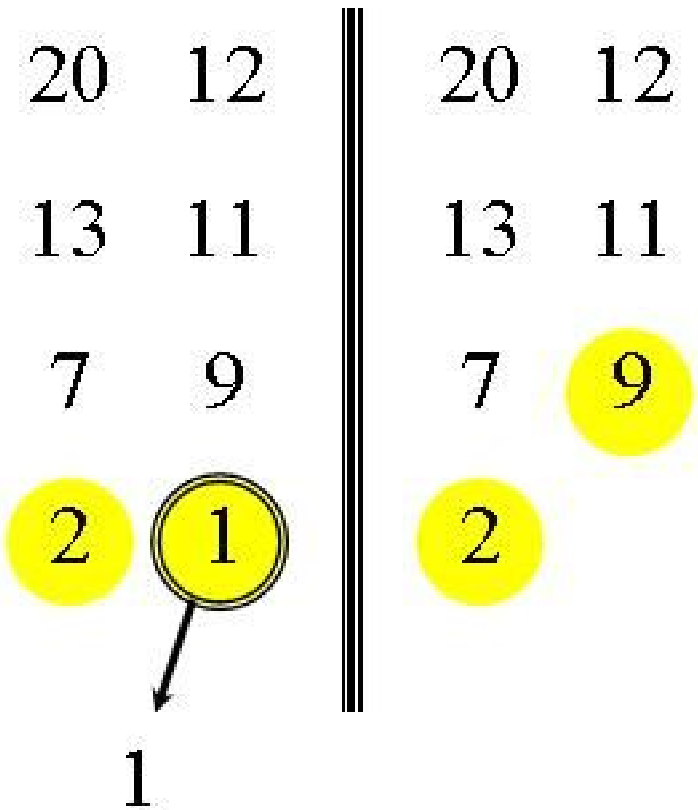

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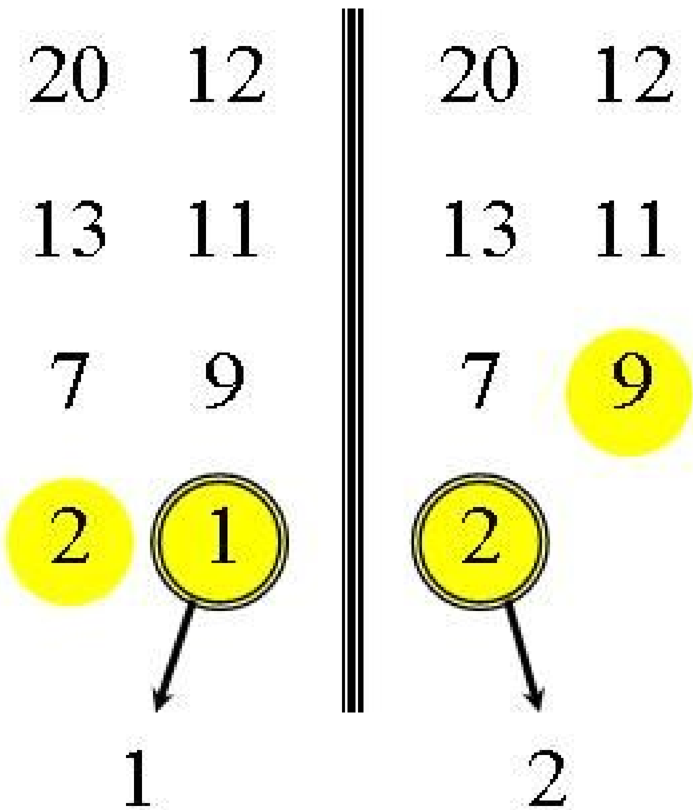

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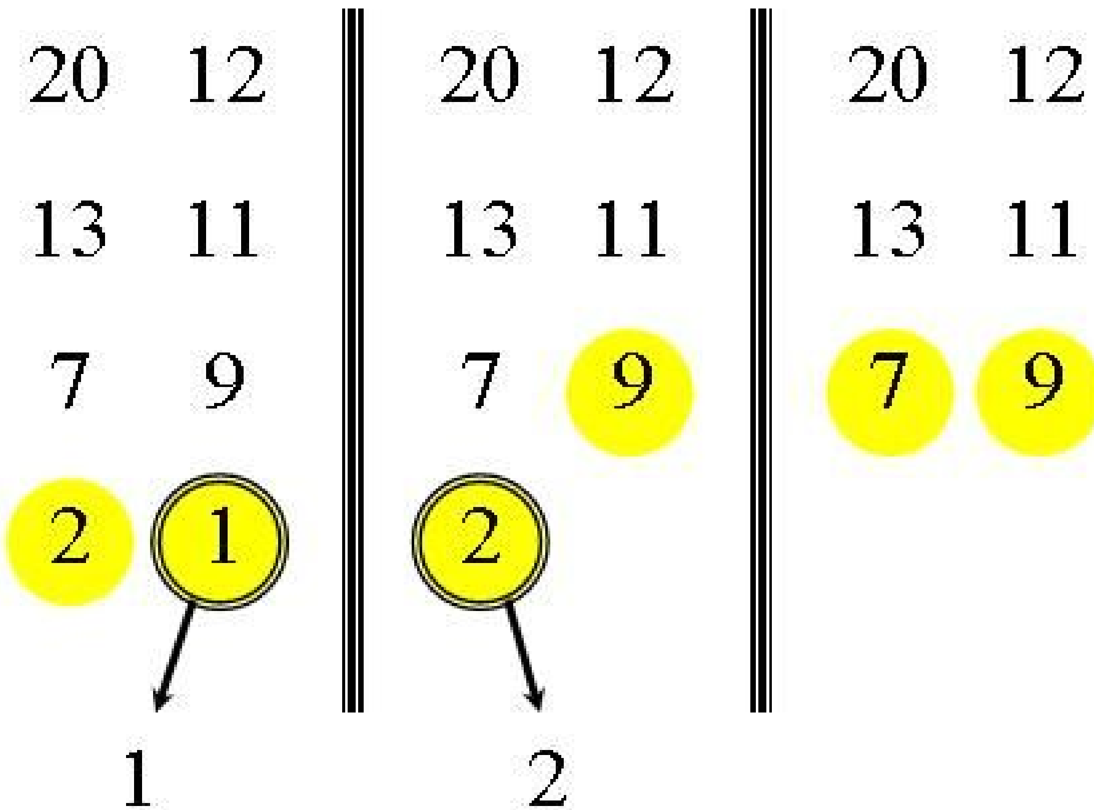

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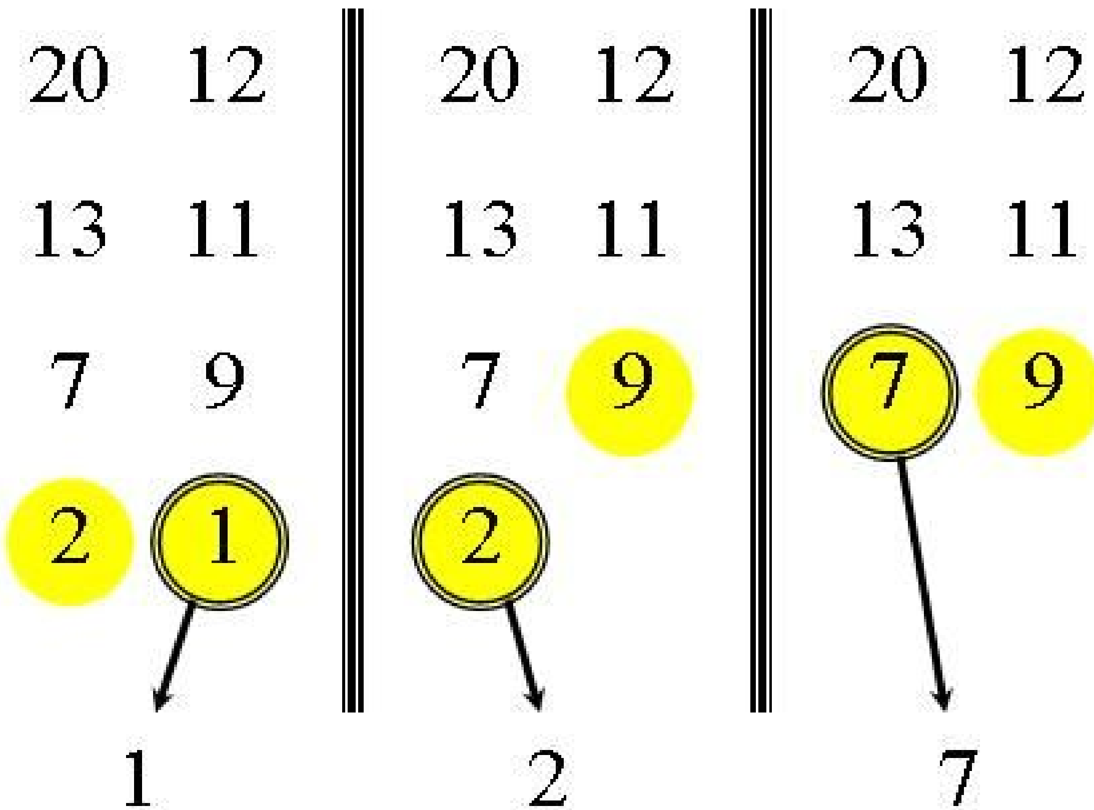

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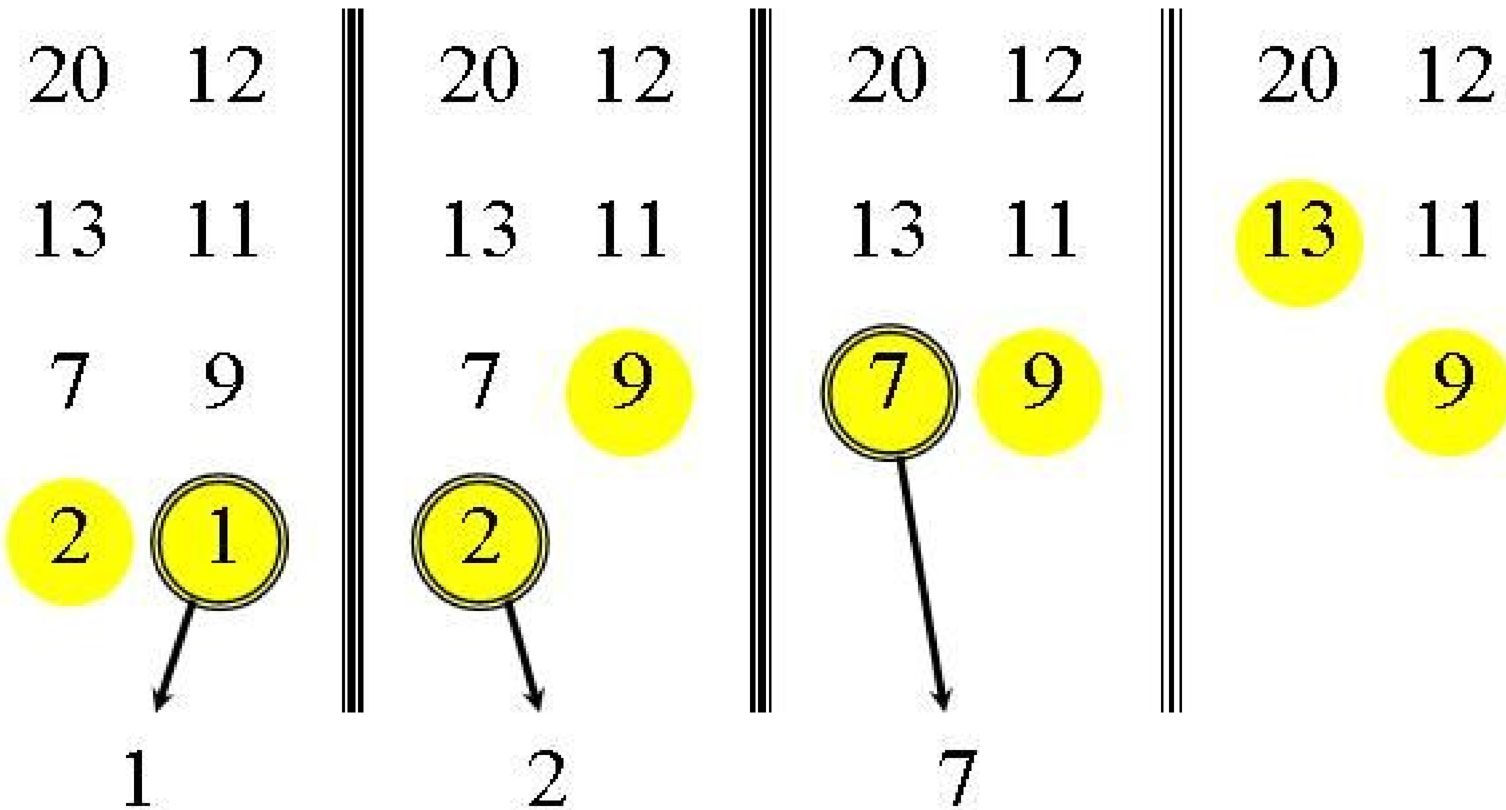

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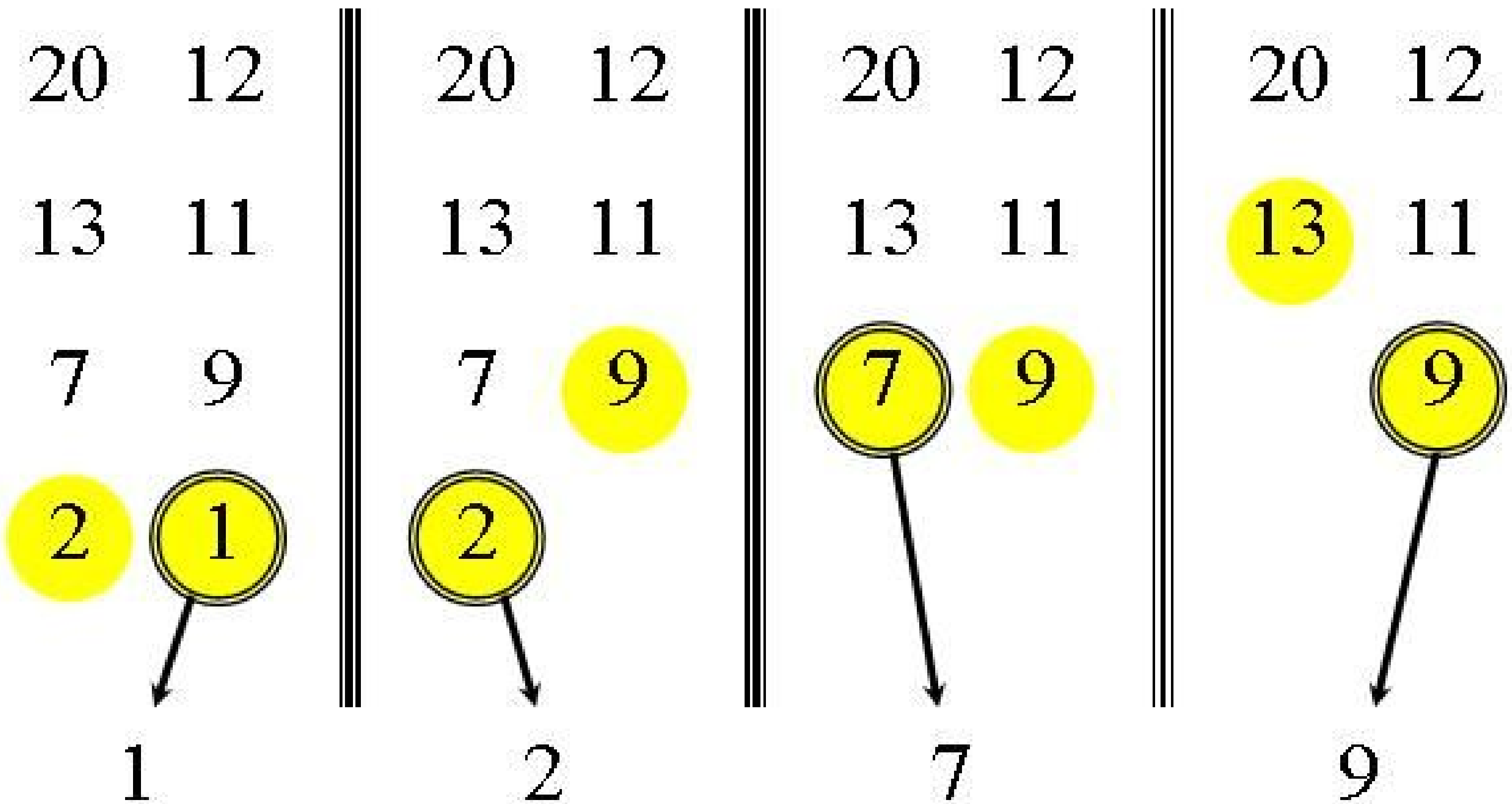

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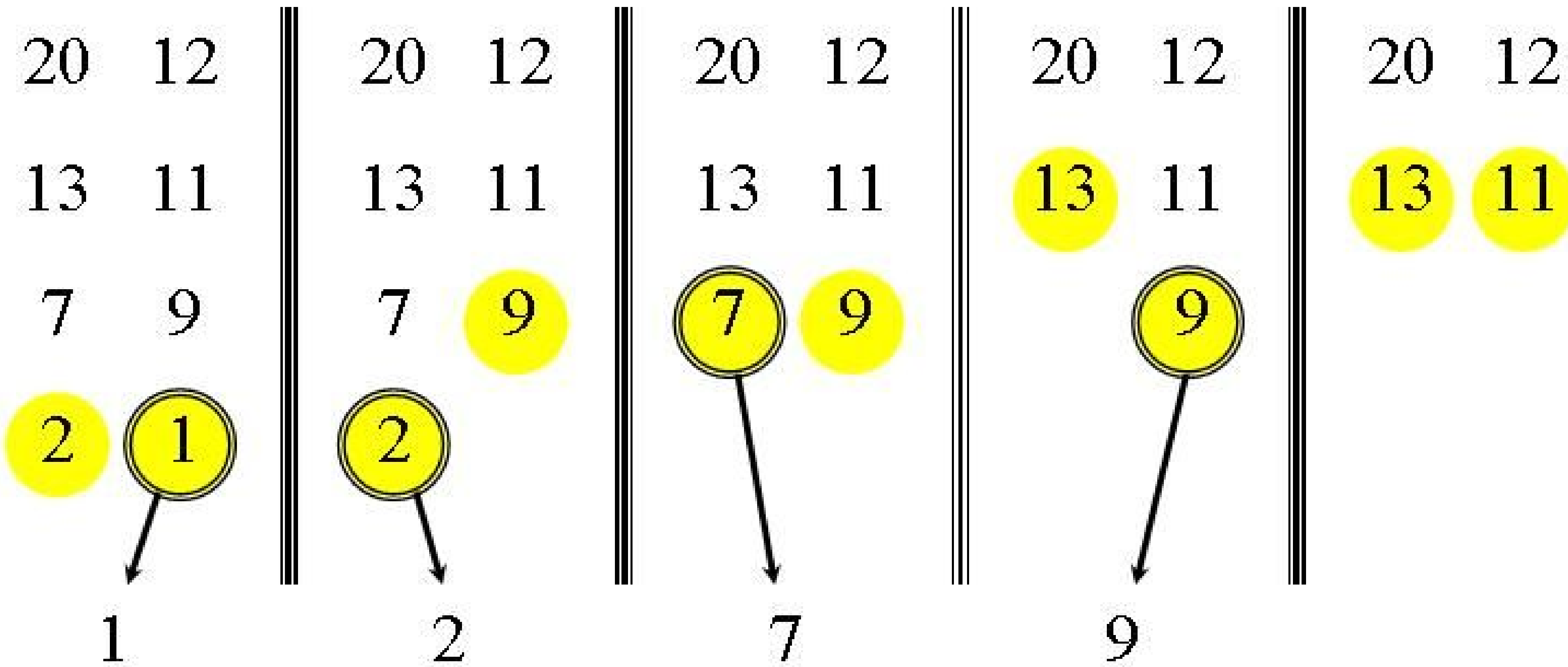

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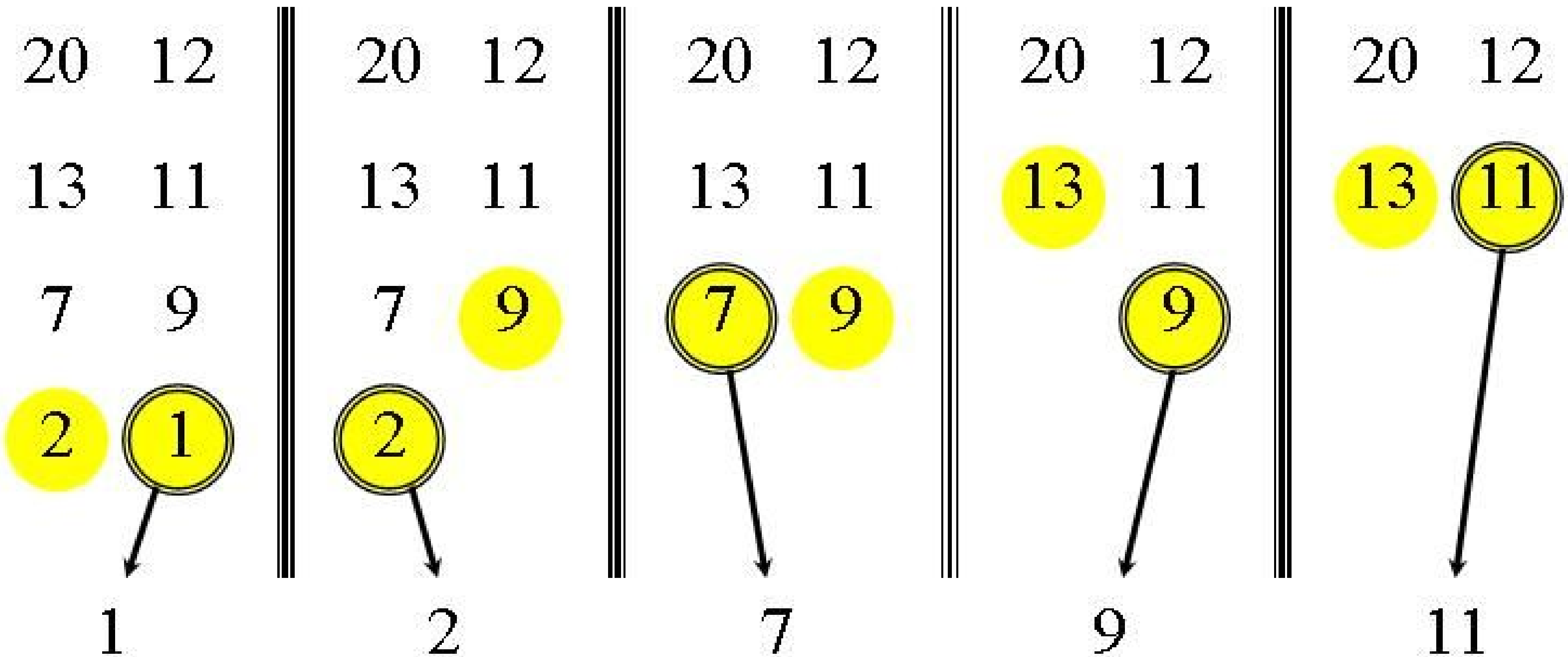

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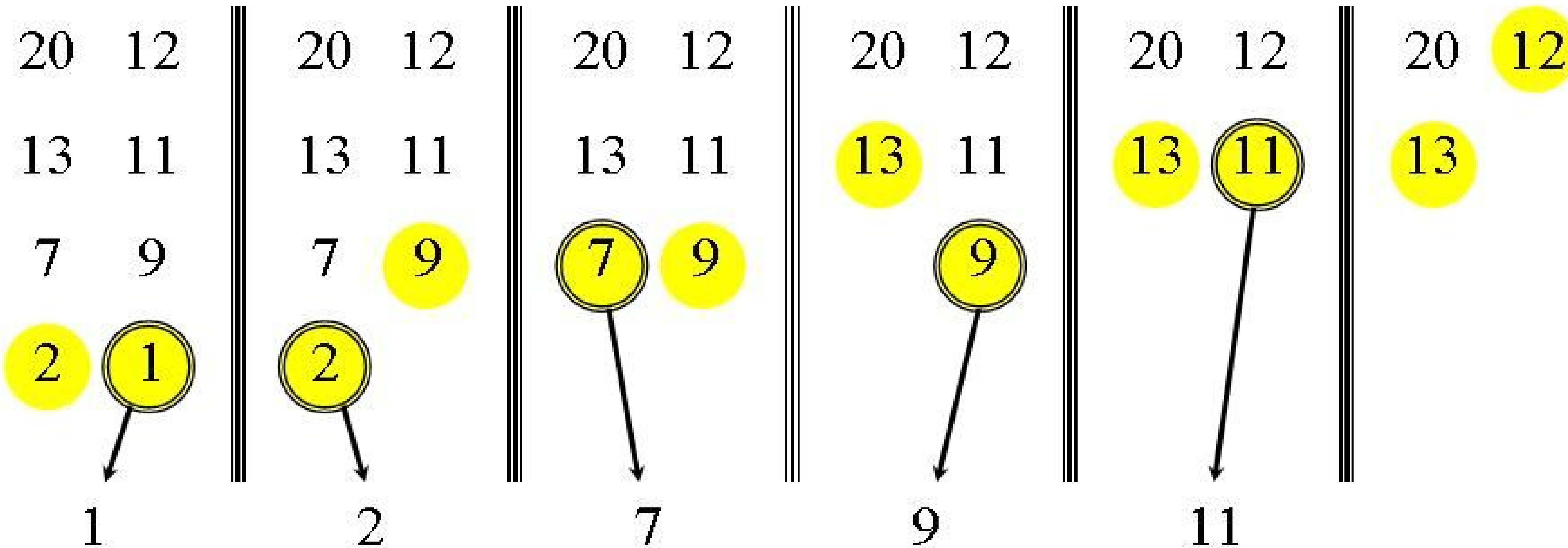

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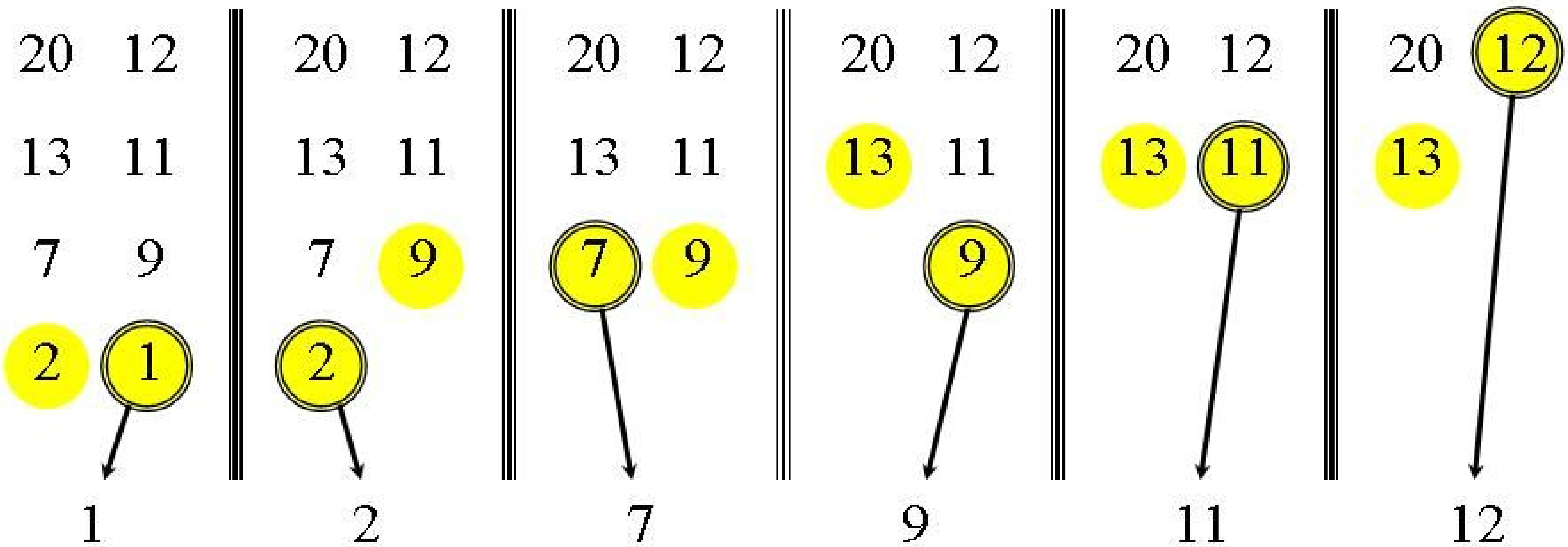

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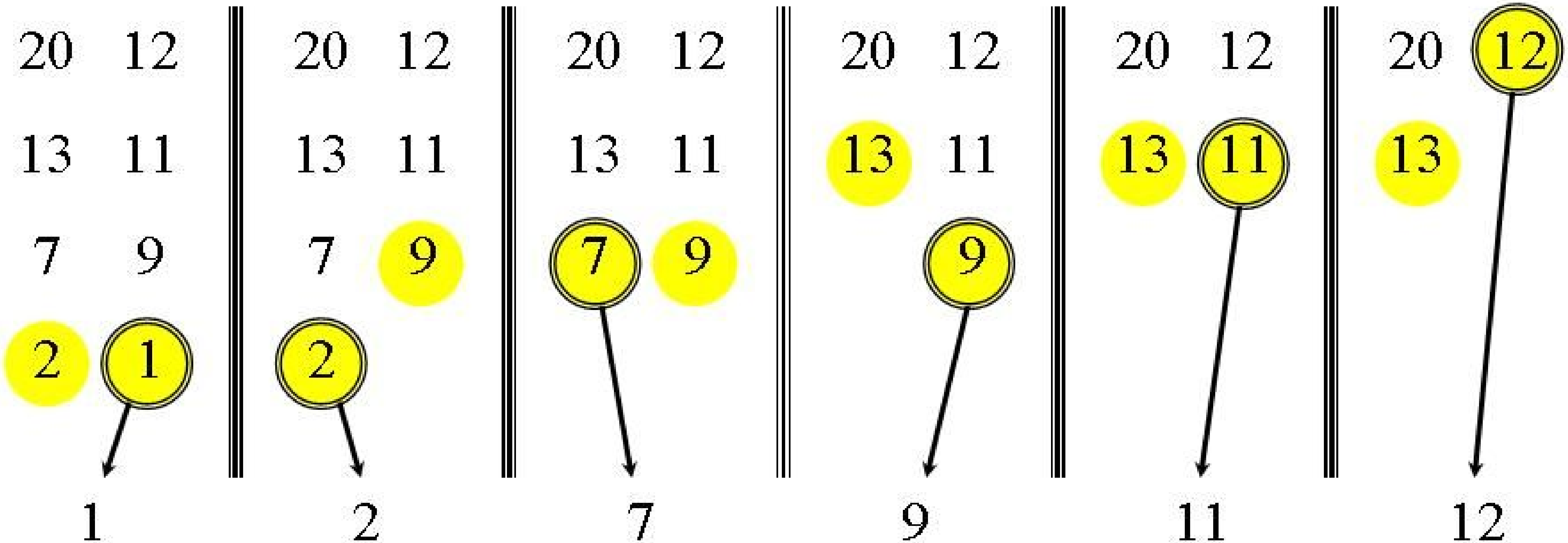


Time = $\Theta(n)$ to merge a total of n elements (linear time).

Analyzing merge sort

MERGE-SORT $A[1 \,..\, n]$	$T(n)$
1. If $n = 1$, done.	$\Theta(1)$
2. Recursively sort $A[\,1 \,..\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,..\, n\,]$.	$2T(n/2)$
3. ***"Merge"*** the two sorted lists	$\Theta(n)$

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

Recurrence solving

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

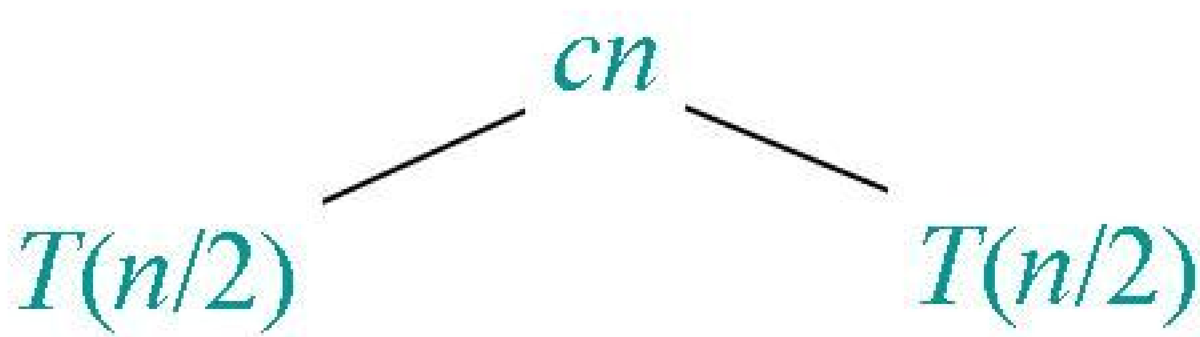
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$T(n)$$

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



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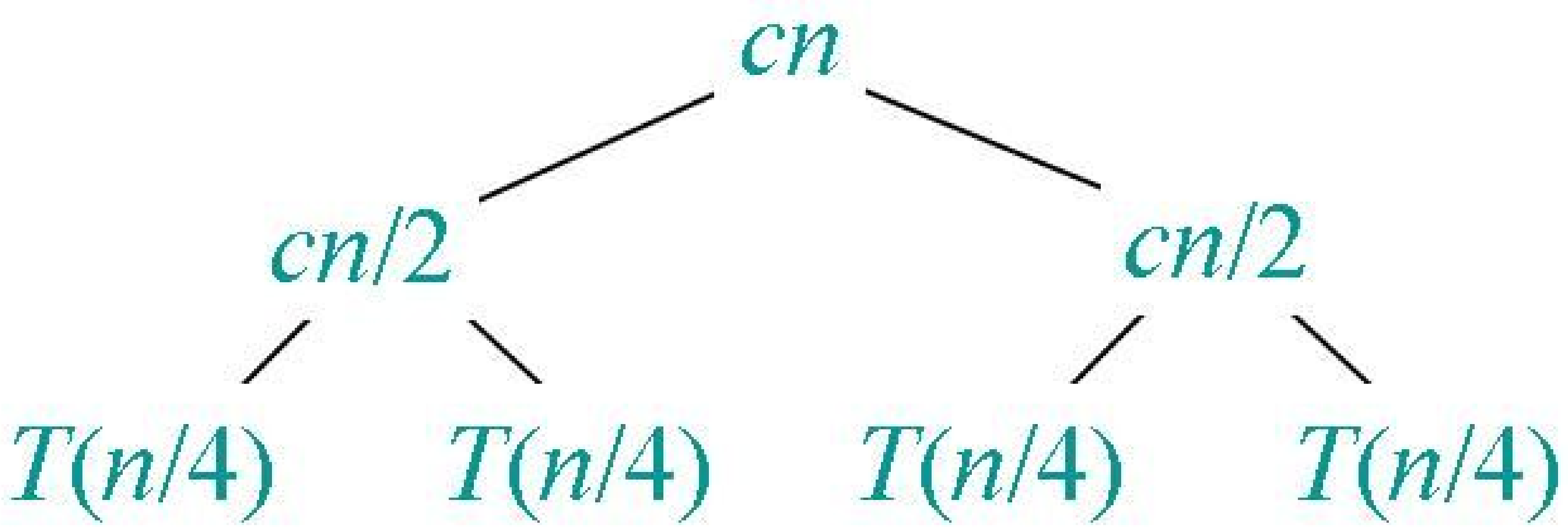

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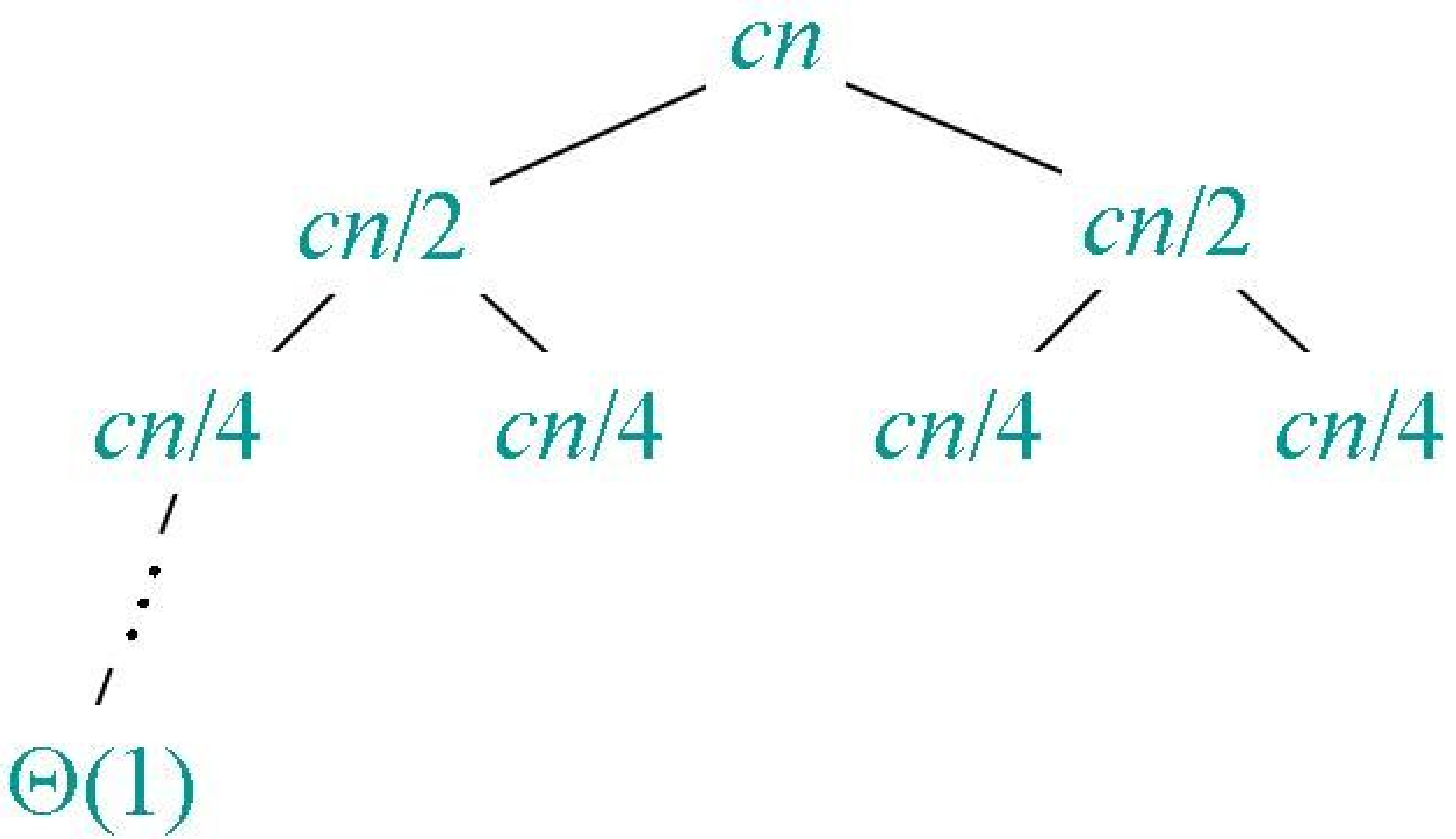

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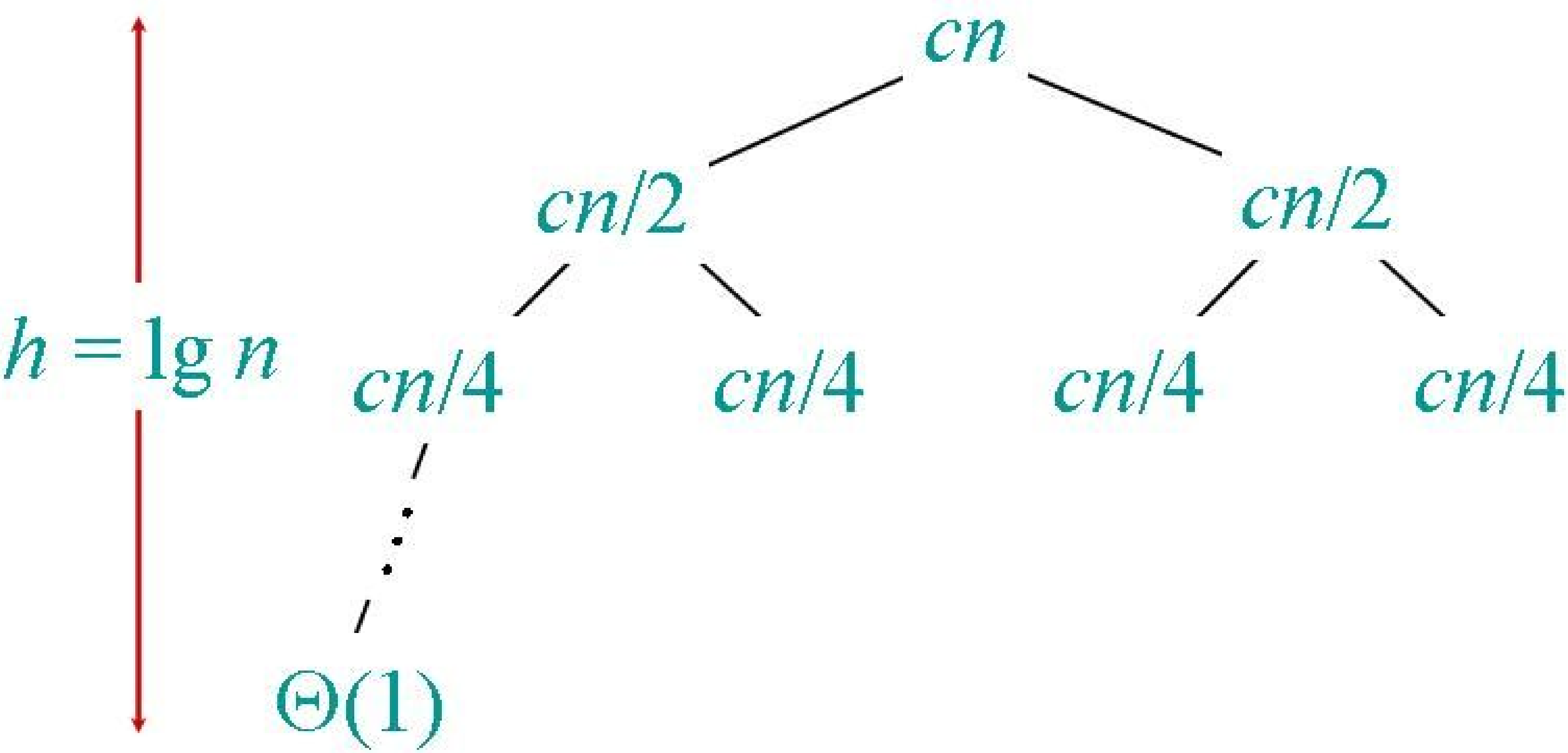

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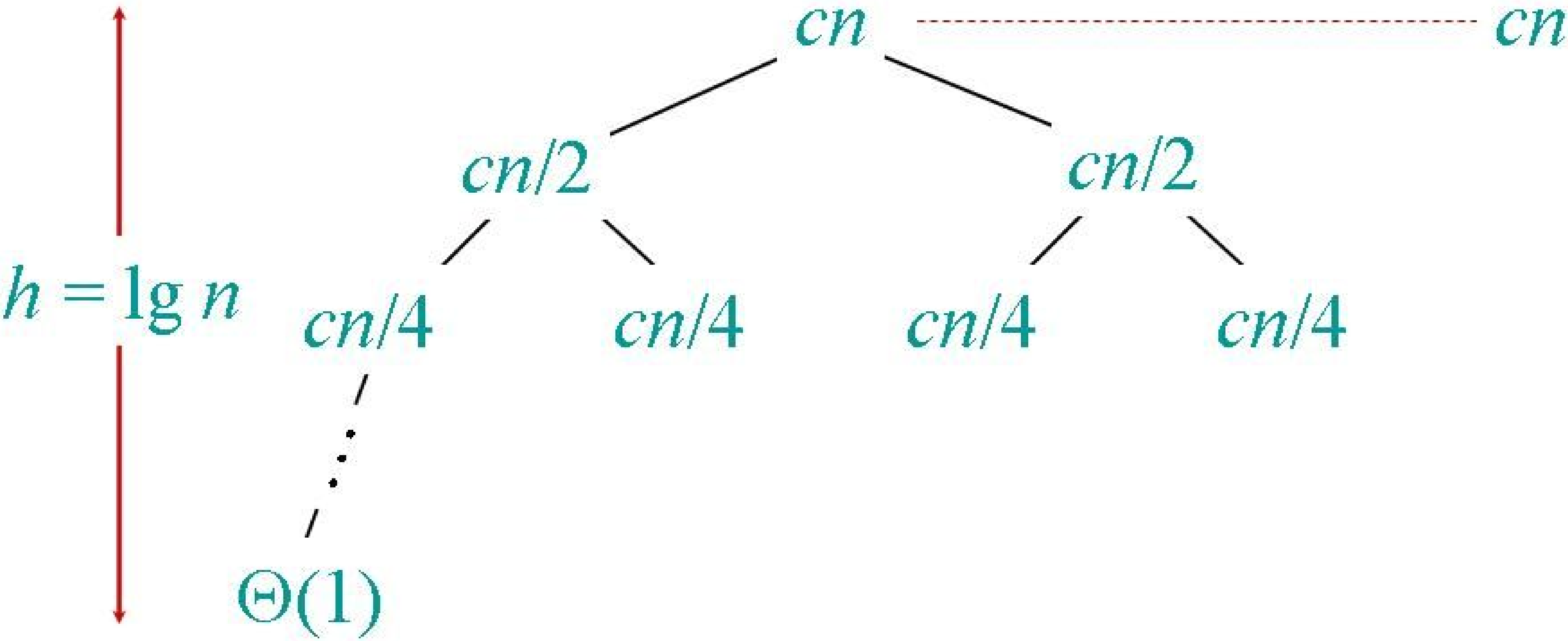

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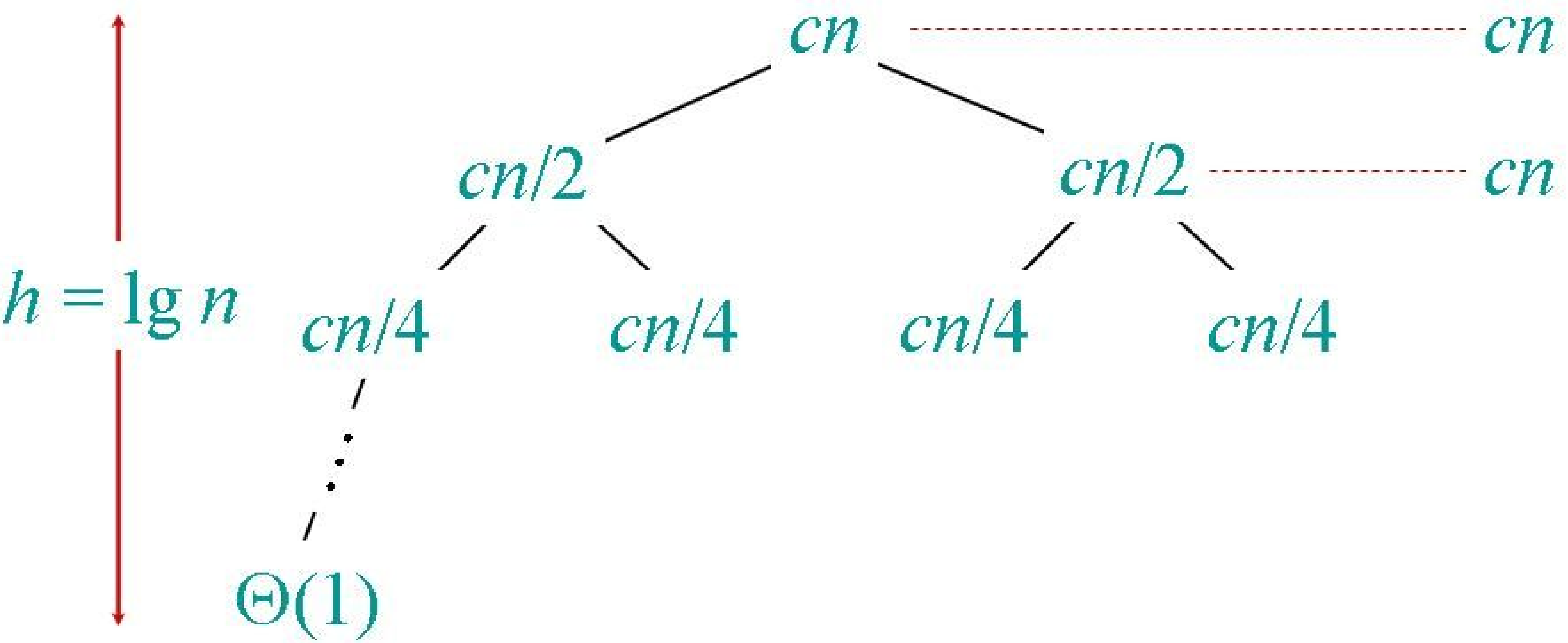

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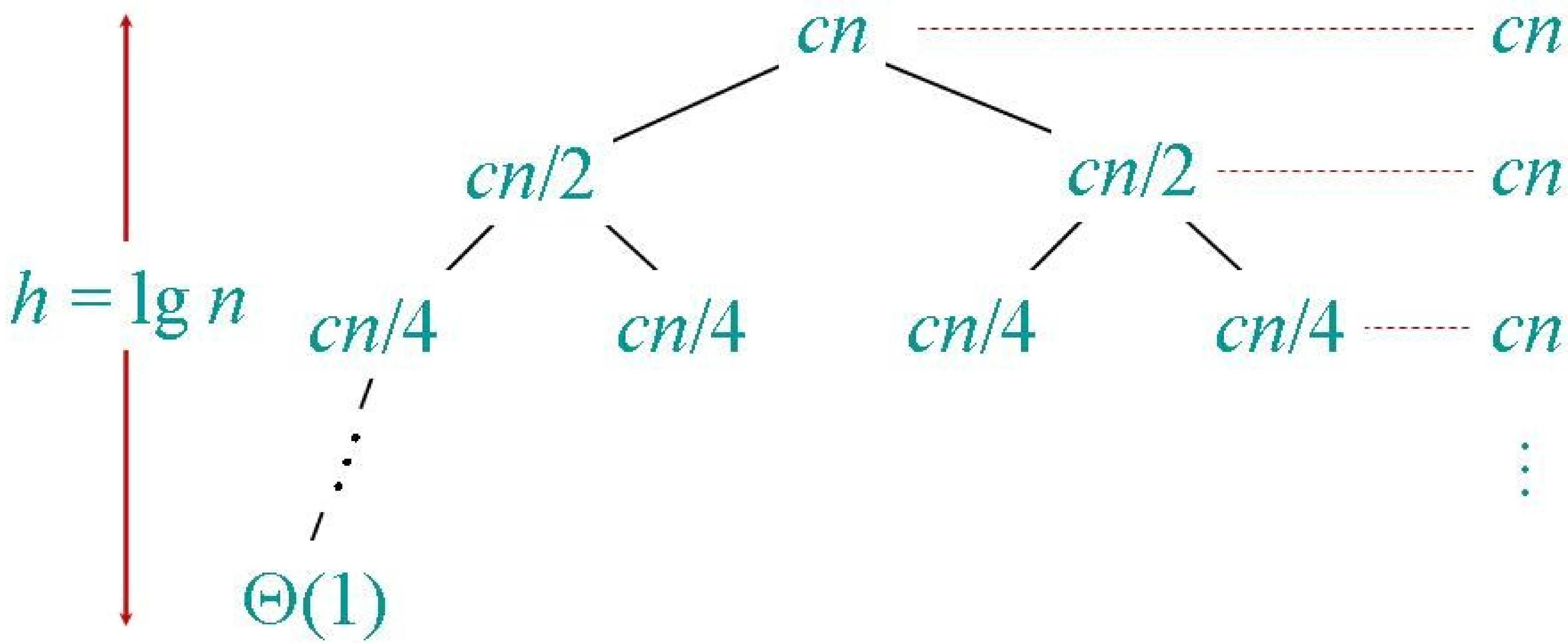

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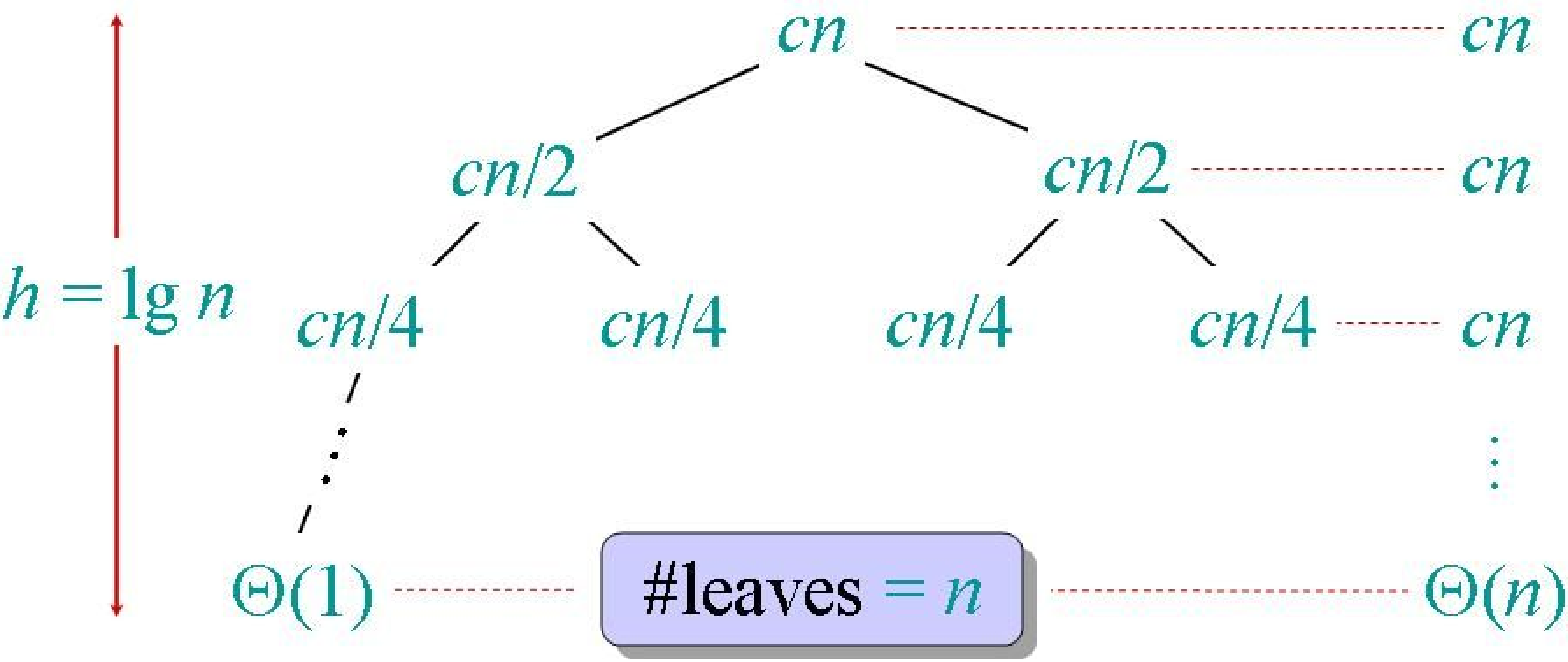

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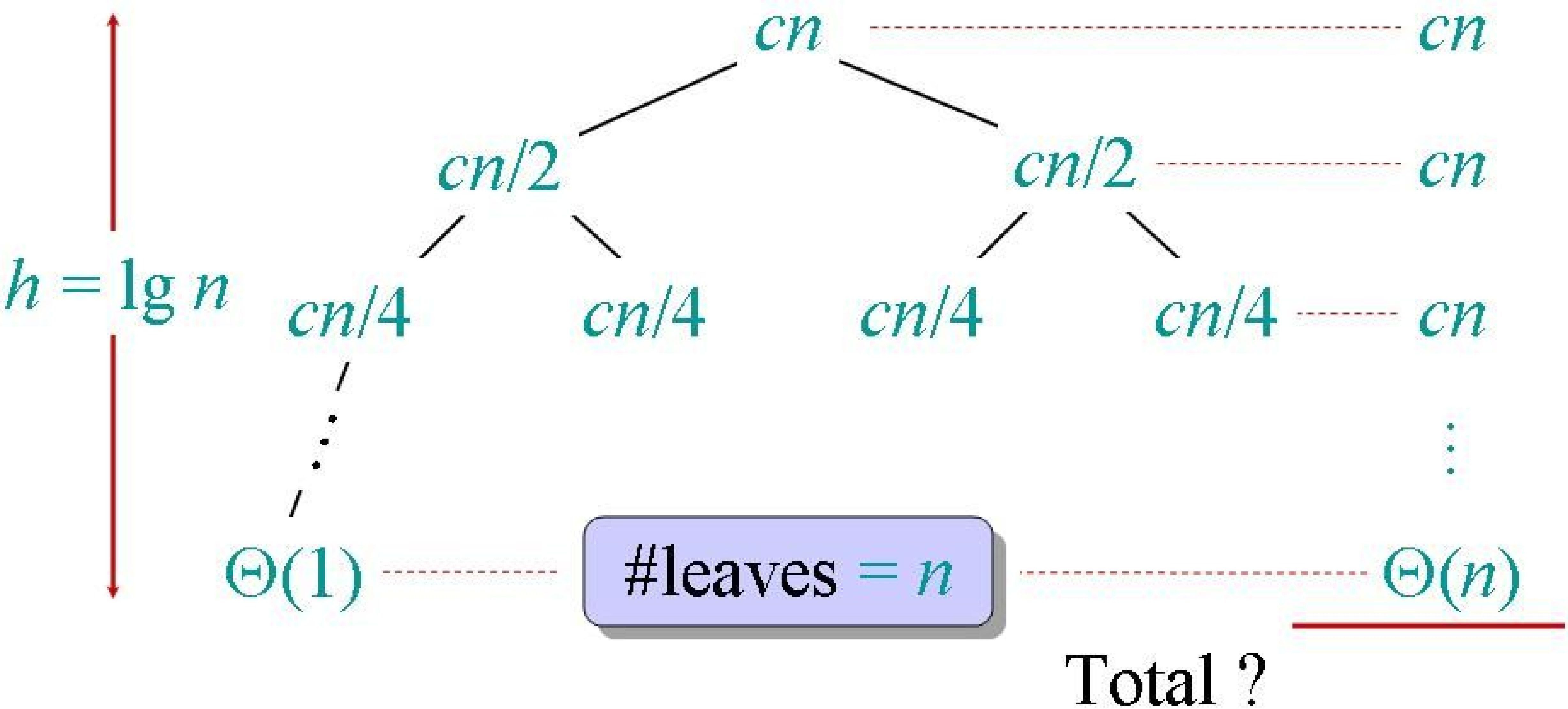

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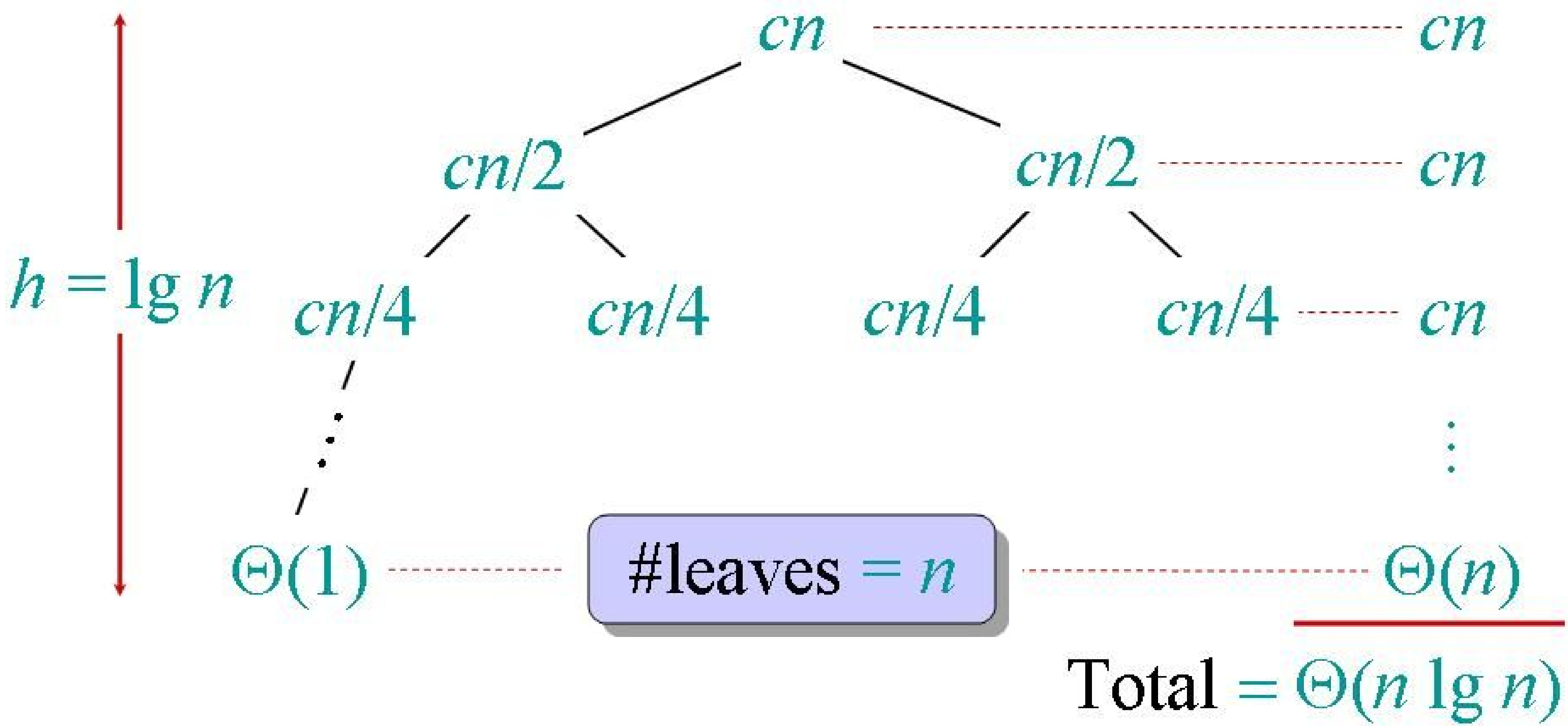


The master method

"One theorem to rule them all" (sort of)

The master method applies to recurrences of the form

#subprobleMs — size of each subproblem — time to split into subproblems and combine results

$$T(n) = a\,T(n/b) + f(n),$$

where $a \geq 1$, $b > 1$, and f is positive.

e.g. Mergesort: $a=2$, $b=2$, $f(n)=O(n)$

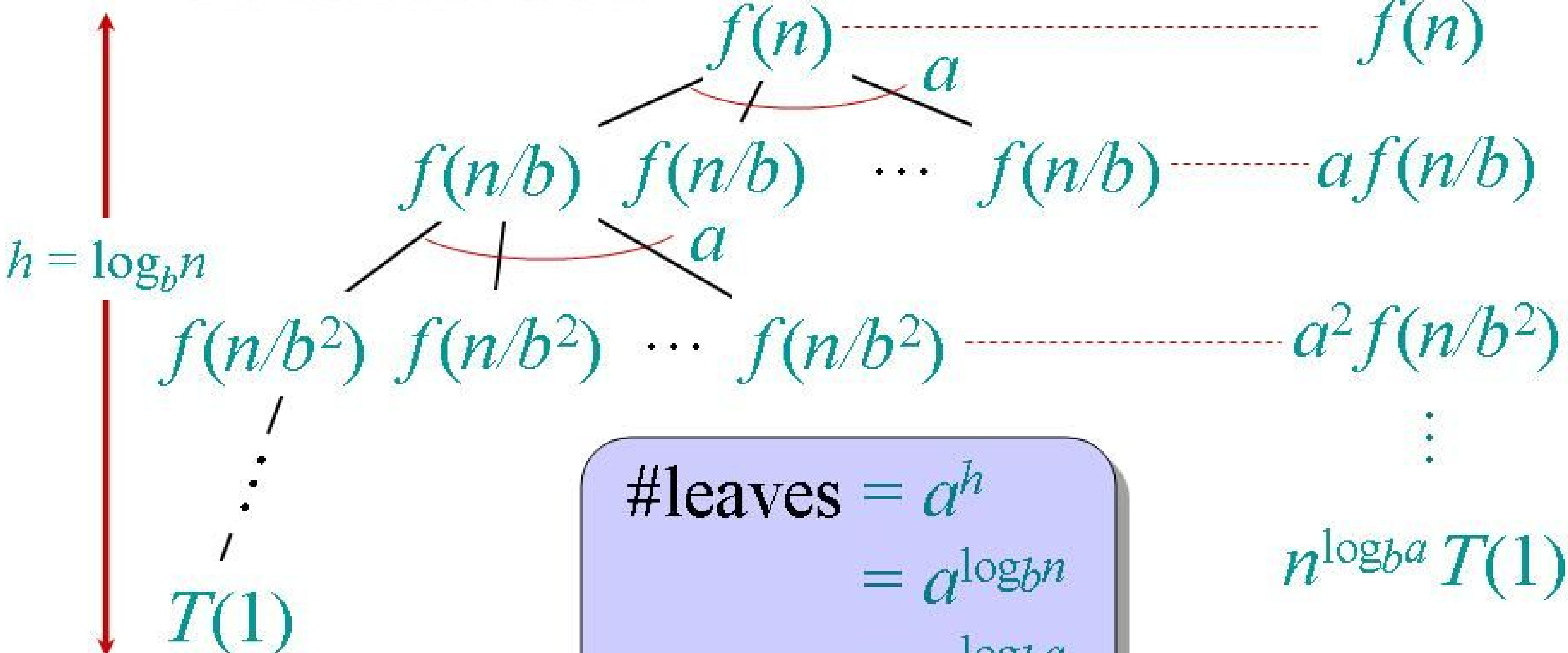
e.g.2 Binary Search: $a=1$, $b=2$, $f(n)=O(1)$

Basic Idea: Compare $f(n)$ with $n^{\log_b a}$.

Idea of master theorem

$T(n) = a\,T(n/b) + f(n)$

Recursion tree:



#leaves $= a^h$

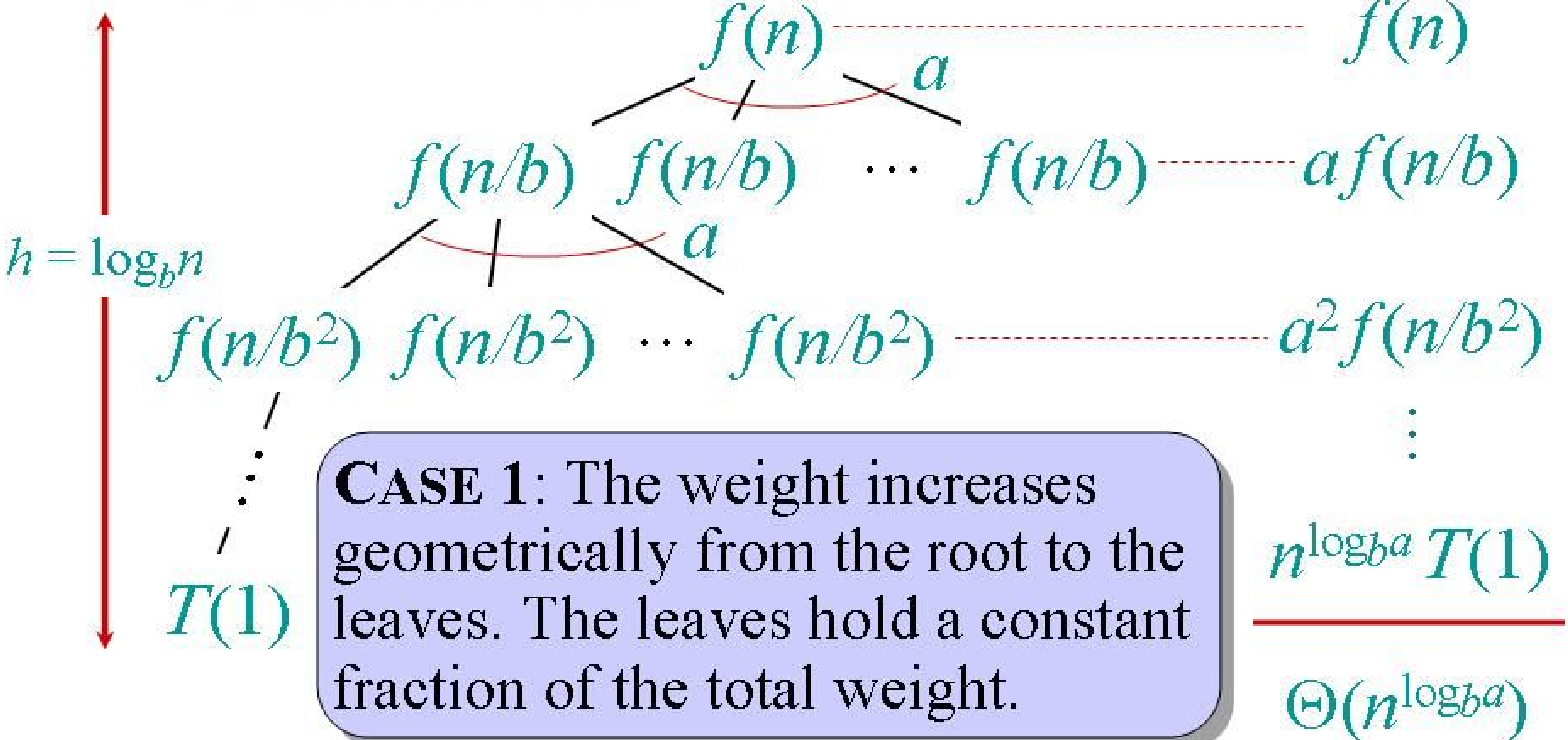
$= a^{\log_b n}$

$= n^{\log_b a}$

Idea of master theorem

$T(n) = a\,T(n/b) + f(n)$

Recursion tree:

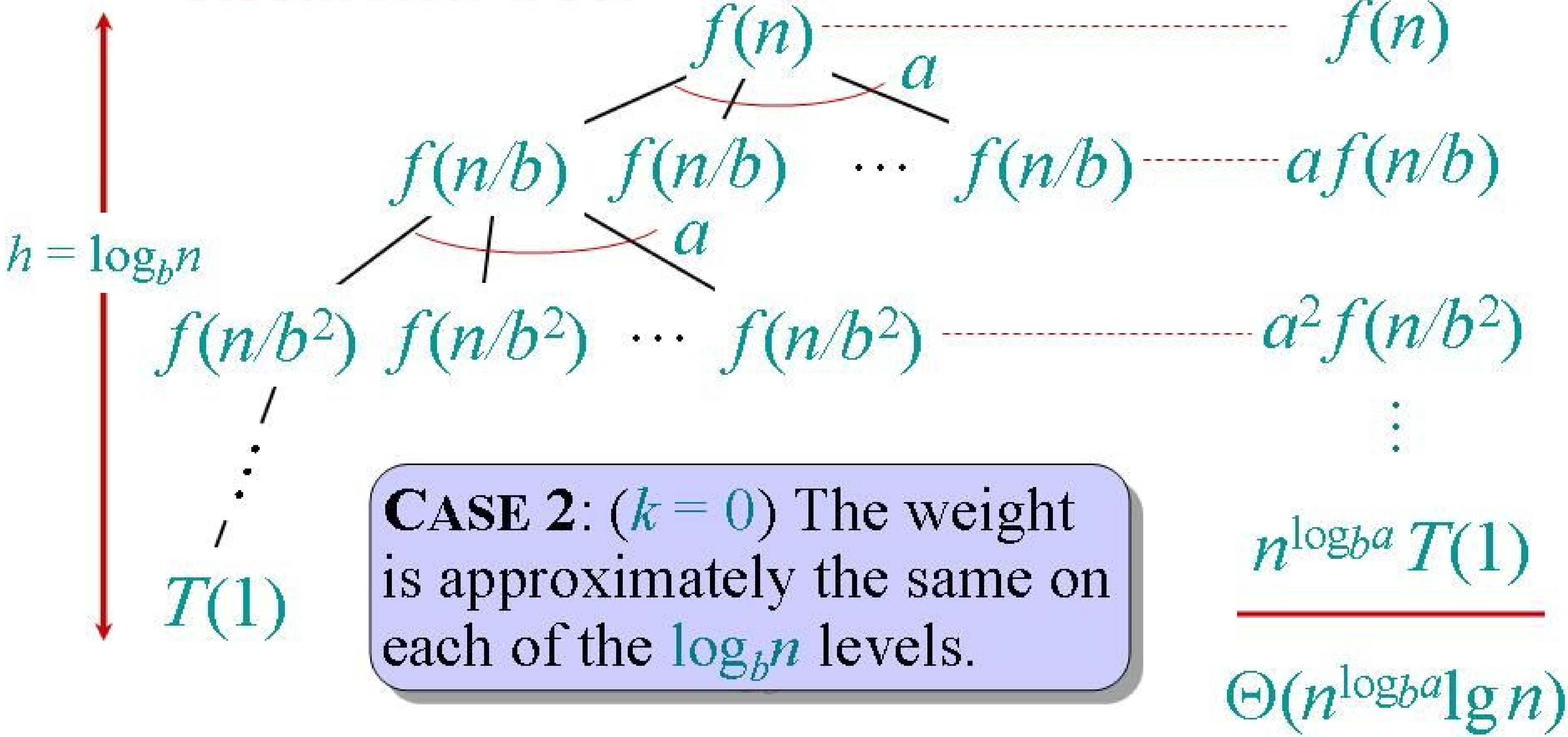


CASE 1: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

Idea of master theorem

$T(n) = a\,T(n/b) + f(n)$

Recursion tree:

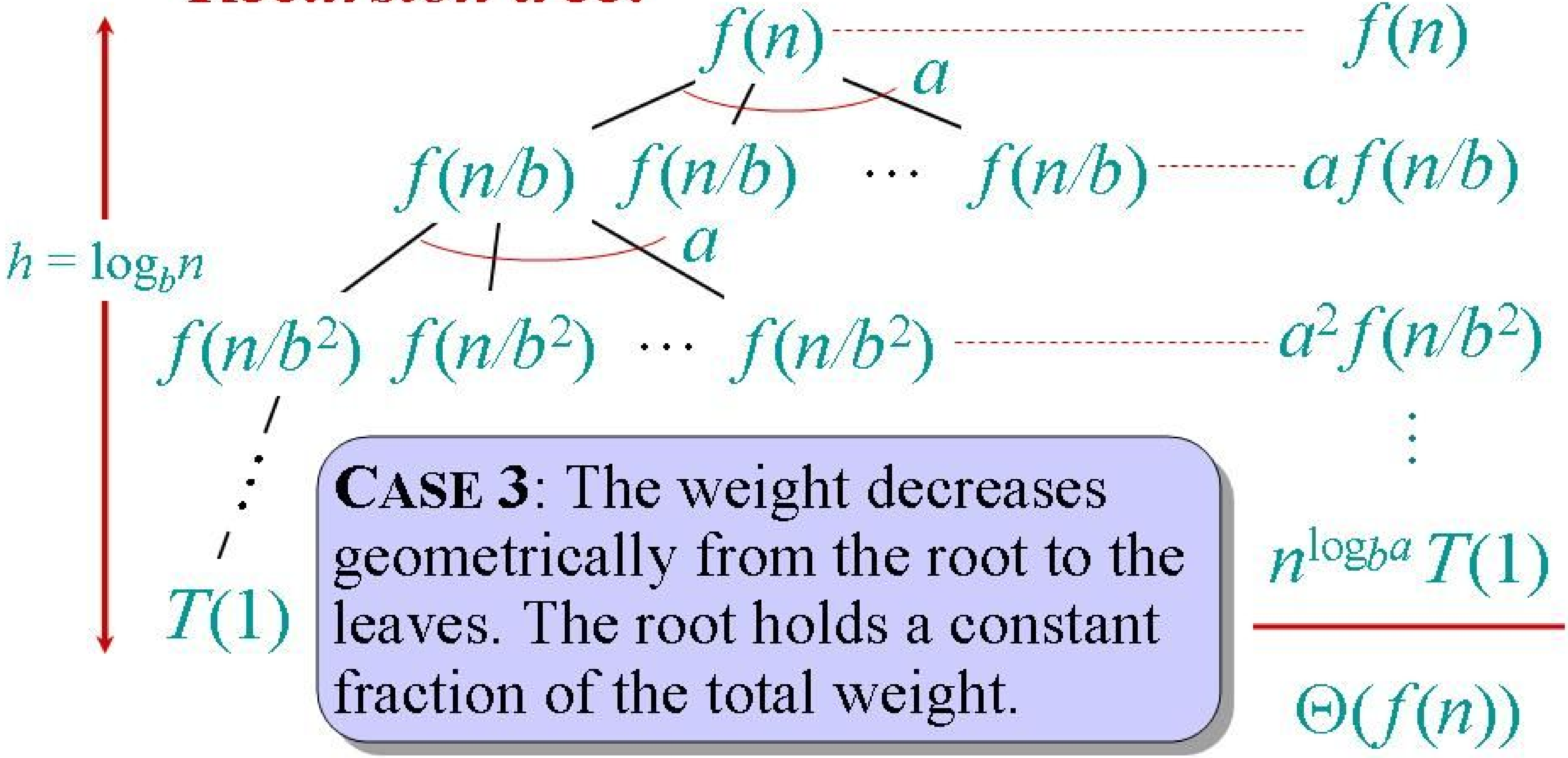


CASE 2: ($k = 0$) The weight is approximately the same on each of the $\log_b n$ levels.

Idea of master theorem

$$T(n) = a\,T(n/b) + f(n)$$

Recursion tree:



CASE 3: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = \Theta(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

     cost of level $i = a^i f(n/b^i) = \Theta(n^{\log_b a-\varepsilon} \cdot b^{i\,\varepsilon})$
     so geometric increase of cost as we go deeper in the tree
     hence, leaf level cost dominates!

   Solution: $T(n) = \Theta(n^{\log_b a})$.

Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

2. $f(n) = \Theta(n^{\log_b a} \log^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

(cost of level i) = $a^i f(n/b^i) = \Theta(n^{\log_b a} \cdot \log^k(n/b^i))$
so all levels have about the same cost

Solution: $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Theta(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor).

     (cost of level i) = $a^i f(n/b^i) = \Theta(n^{\log_b a+\varepsilon} \cdot b^{-i\,\varepsilon})$

     so geometric decrease of cost as we go deeper in the tree

     hence, root cost dominates!

Solution: $T(n) = \Theta(f(n))$.

Examples

Ex. $T(n) = 2T(n/2) + 1$

$a = 2, b = 2 \Rightarrow n^{\log_b a} = n; f(n) = 1.$

Case 1: $f(n) = O(n^{1-\varepsilon})$ for $\varepsilon = 1$.

$\therefore T(n) = \Theta(n).$

Ex. $T(n) = 2T(n/2) + n$

$a = 2, b = 2 \Rightarrow n^{\log_b a} = n; f(n) = n.$

Case 2: $f(n) = \Theta(n \lg^0 n)$, that is, $k = 0$.

$\therefore T(n) = \Theta(n \lg n).$

Examples

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^3.$

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^3)$.