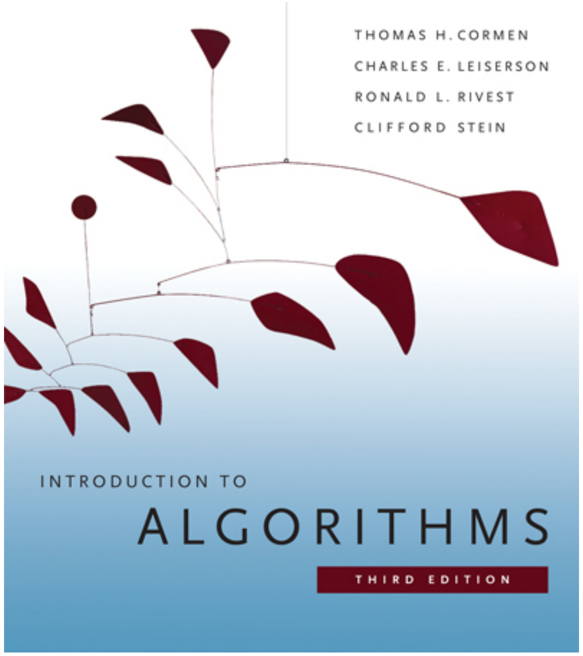
6.006- *Introduction to Algorithms*



Lecture 7

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CLRS: 11.4, 17.

So Far

- Hash table as dictionary
  - Insert/Search/Delete
- Collisions by chaining
  - Build a linked list in each bucket
  - Operation time is length of list
- Simple Uniform Hashing
  - Every item to uniform random bucket
  - n items in size m table → average length $n/m = \alpha$
- Signatures: fast comparison
- Rolling Hash: fast sequence of hash's

DYNAMIC DICTIONARIES

Dynamic Dictionaries

- In substring application, inserted all at once then scanned
- More generally, arbitrary sequence of insert, delete, find
- How do we know how big the table will get?
- What if we guess wrong?

  too small ➔ load high, operations too slow

  too large ➔ high initialization cost, consumes space, potentially more cache-misses
- Want $m=\Theta(n)$ at all times

Solution: Resize

- Start table at small constant size
- When table too full, make it bigger
- When table too empty, make it smaller
- How?
  - Build a whole new hash table and insert items
  - Recompute all hashes
  - Recreate new linked lists
  - Time spent to rebuild:
    (new-size) + #hashes x (HashTime)

When to resize?

- **Approach 1:** whenever $n > m$, rebuild table to new size
  - Sequence of n inserts
  - Each increases n past m, causes rebuild
  - Total work: $\Theta(1 + 2 + \ldots + n) = \Theta(n^2)$
- **Approach 2:** Whenever $n \geq 2m$, rebuild table to new size
  - ***Costly inserts*****:** insert 2^i for all i:
    These cost: $\Theta(1 + 2 + 4 + \ldots + n) = \Theta(n)$
  - All other inserts take O(1) time – ***why?***
  - Inserting n items takes O(n) time
  - Keeps m a power of 2 --- good for mod

a factor of (HashTime) is suppressed here

Amortized Analysis

- If a sequence of n operations takes time T, then each operation has amortized cost T/n
  - Like amortizing a loan: payment per month
- Rebuilding when $n \geq 2m$ → some ops are very slow
  - $\Theta(n)$ for insertion that causes last resize
- But on average, fast
  - $O(1)$ amortized cost per operation
- Often, only care about total runtime
  - So averaging is fine

Insertions+Deletions?

- Rebuild table to new size when $n < m$?
  - Same as bad insert: $O(n^2)$ work
- Rebuild when $n<m/2$?
  - Makes a sequence of deletes fast
  - What about an arbitrary sequence of inserts/deletes?
    - Suppose we have just rebuilt: $m=n$
    - Next rebuild a grow → at least m more inserts are needed before growing table
      - Amortized cost $O(2m / m)) = O(1)$
    - Next rebuild a shrink → at least $m/2$ more deletes are needed before shrinking
      - Amortized cost $O(m/2 / (m/2)) = O(1)$

Another Approach

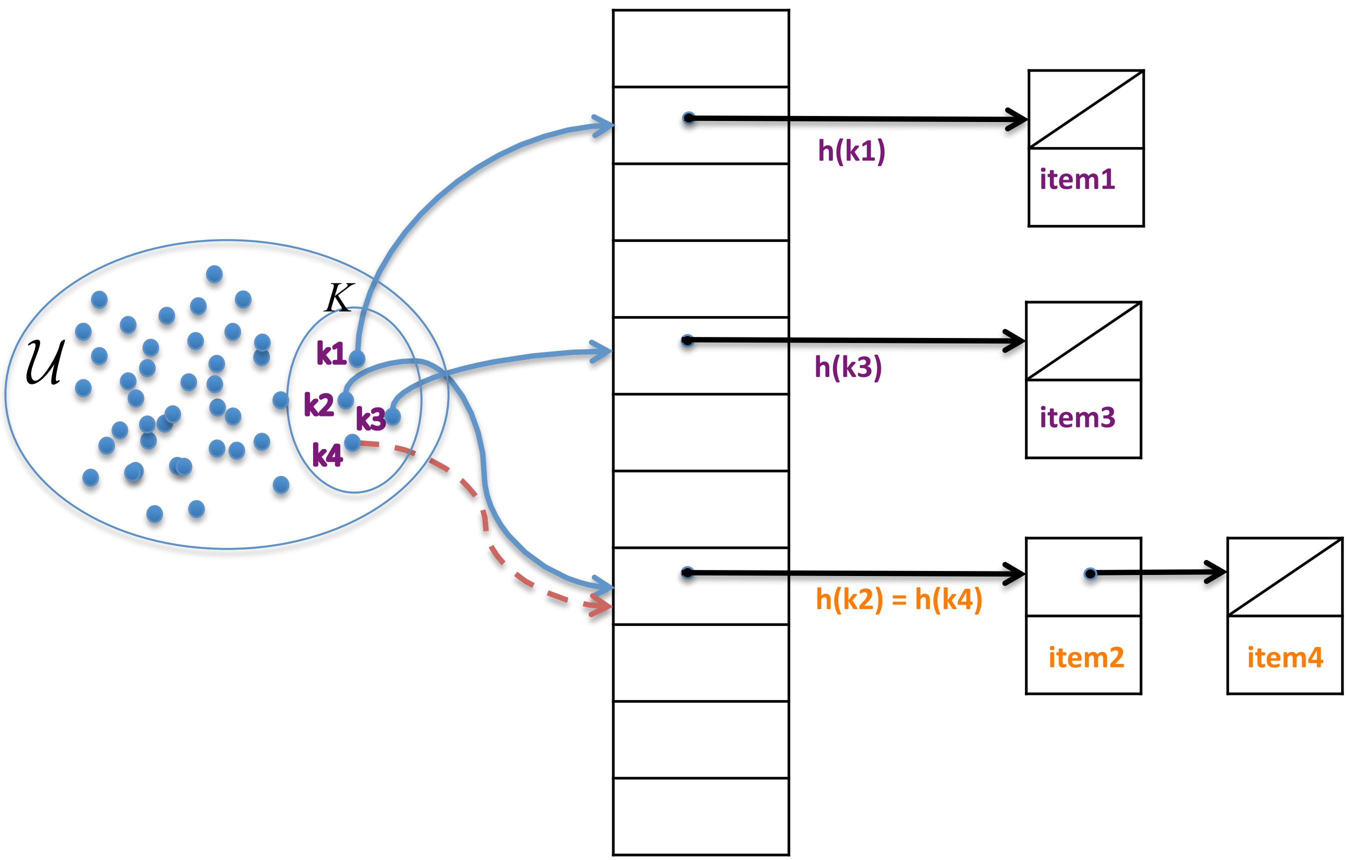
- Algorithm
  - Keep m a power of 2 (good for mod)
  - Grow (double m) when $n \geq m$
  - Shrink (halve m) when $n \leq m/4$
- Analysis
  - Just after rebuild: $n=m/2$
  - Next rebuild a grow → at least m/2 more inserts
    - Amortized cost $O(2m / (m/2)) = O(1)$
  - Next rebuild a shrink → at least m/4 more deletes
    - Amortized cost $O(m/2 / (m/4)) = O(1)$

Summary

- Arbitrary sequence of insert/delete/find
- O(1) amortized time per operation

OPEN ADDRESSING

Recall Chaining…



$\mathcal{U}$: *universe of all possible keys-huge set*

K : *actual keys-small set, but not known when designing data structure*

Open Addressing

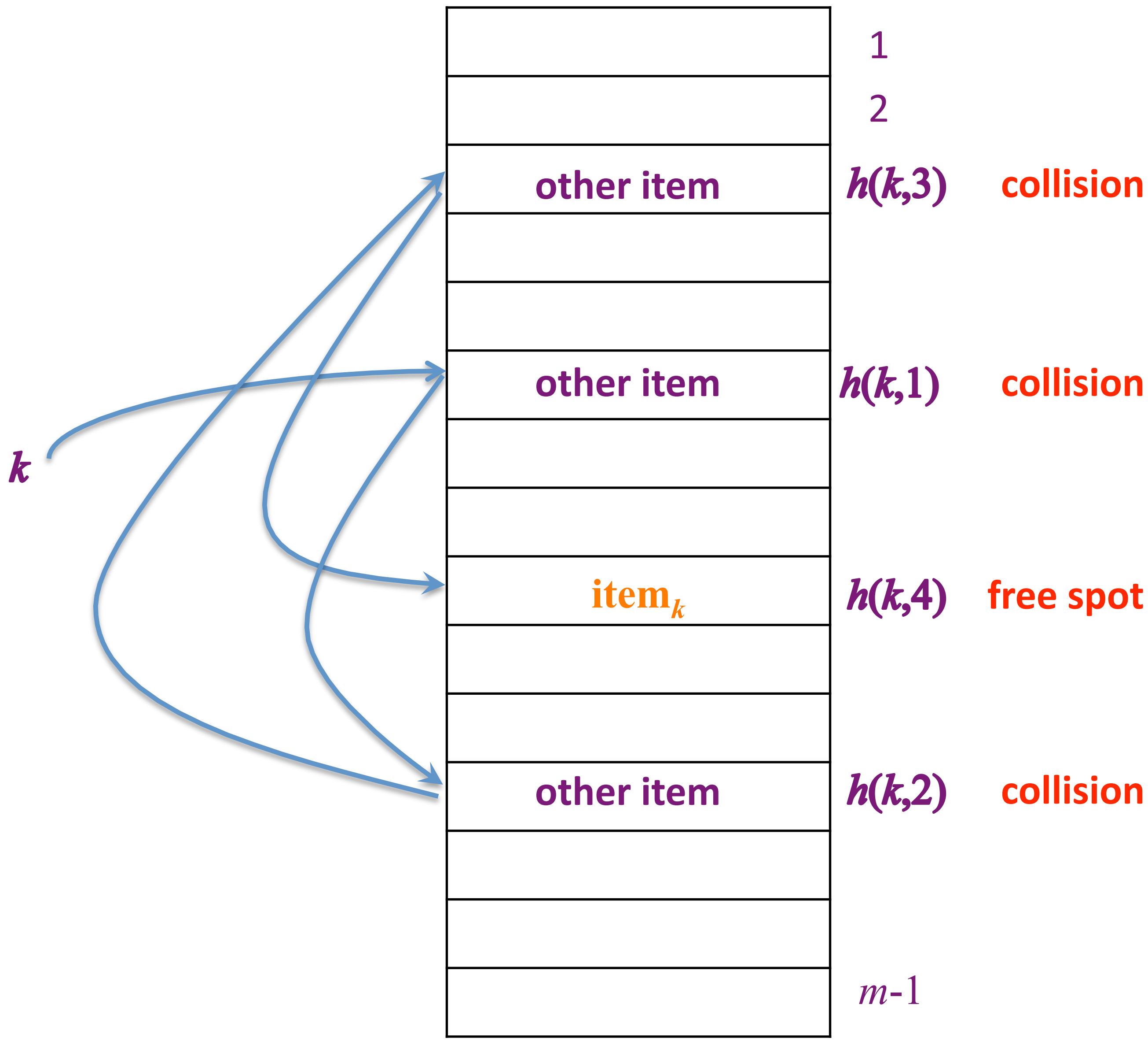
- Different technique for dealing with collisions; does not use linked list
- Instead: if bucket occupied, find other bucket (need $m \geq n$)
- For insert: probe a sequence of buckets until find empty one!
- $h(x)$ specifies probe sequence for item x
  - Ideally, sequence visits all buckets
  - $h: U \times [1..m] \rightarrow [1..m]$

Bucket

Universe of keys

Probe number

Open Addressing (example)



Operations

- Insert
  - Probe till find empty bucket, put item there
- Search
  - Probe till find item (return with success)
  - Or find empty bucket (return with failure)
    - Because if item inserted, would use that empty bucket
- Delete
  - Probe till find item
  - Remove~~, leaving empty~~ bucket

Problem with Deletion

- Consider a sequence
  - Insert x
  - Insert y
    - suppose probe sequence for y passes x bucket
    - store y elsewhere
  - Delete x (leaving hole)
  - Search for y
    - Probe sequence hits x bucket
    - Bucket now empty
    - Conclude y not in table (else y would be there)

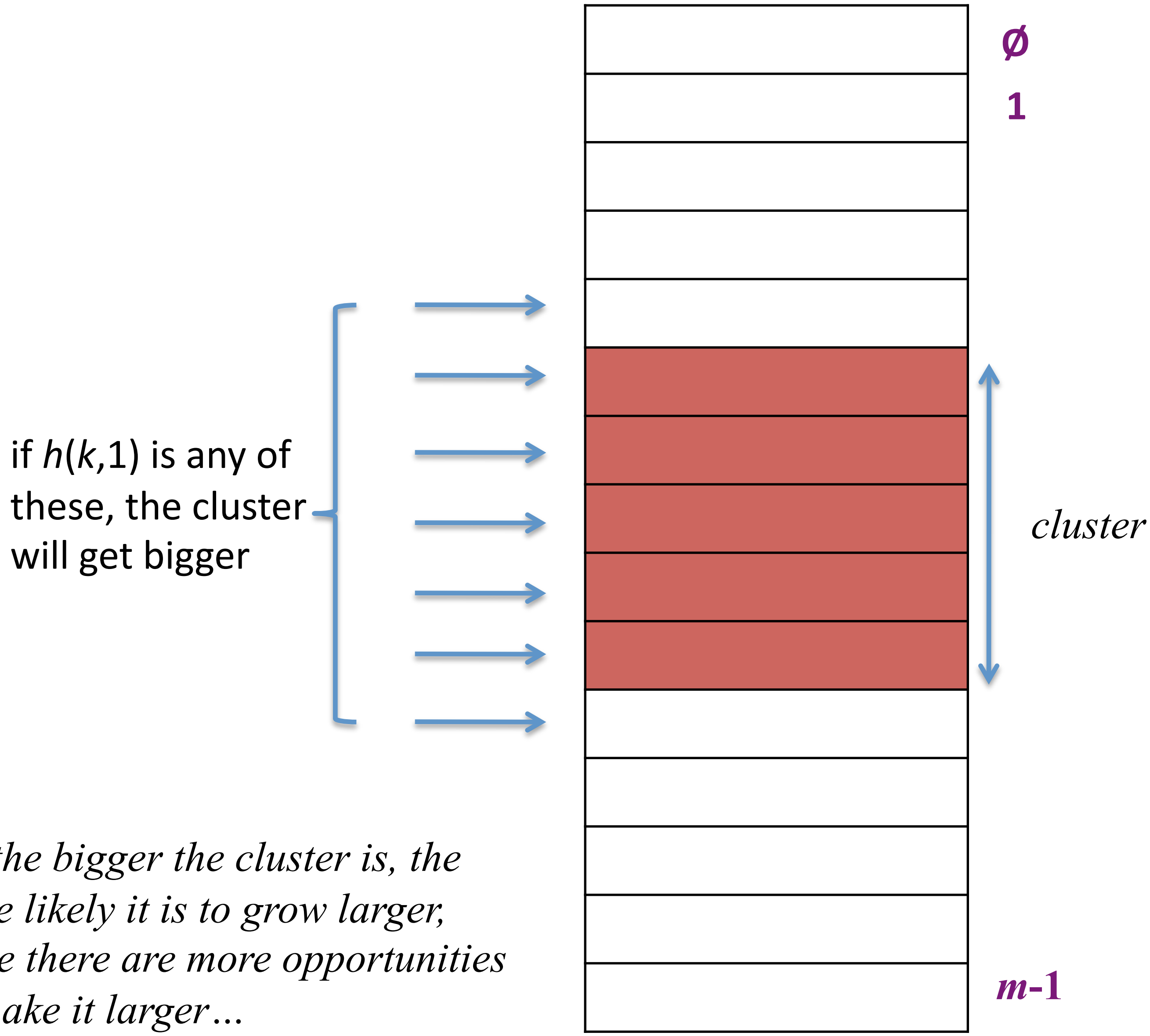
Solution for deletion

- When delete x
  - Leave it in bucket
  - But mark it deleted --- store “tombstone”
- Future search for x sees x is deleted
  - Returns “x not found”
- “Insert z” probes may hit x bucket
  - Since x is deleted, overwrite with z
  - So keeping deleted items doesn’t waste space

What probe sequence?

Linear probing

- $h(k,i) = h'(k) + i$ for ordinary hash h'
- Problem: creates “clusters”, i.e. sequences of full buckets
  - exactly like parking
  - Big clusters are hit by lots of new items
  - They get put at end of cluster
  - Big cluster gets bigger: “rich get richer” phenomenon



i.e. the bigger the cluster is, the more likely it is to grow larger, since there are more opportunities to make it larger…

Linear probing

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- For $0.1 < \alpha < 0.99$, cluster size $\Theta(\log n)$
- Wrecks our constant-time operations

Double Hashing

- Two ordinary hash functions $f(k)$, $g(k)$
- Probe sequence $h(k,i) = f(k) + i \cdot g(k) \bmod m$
- If $g(k)$ relatively prime to m, hits all buckets
  - E.g., if $m=2^r$, make $g(k)$ odd
  - The same bucket is hit twice if for some i,j:

    $f(k) + i \cdot g(k) = f(k) + j \cdot g(k) \bmod m$

    $\rightarrow i \cdot g(k) = j \cdot g(k) \pmod m$

    $\rightarrow (i-j) \cdot g(k) = 0 \pmod m$

    $\rightarrow$ m and $g(k)$ not relatively prime

(otherwise m should divide i-j, which is not possible for $i, j<m$)

Performance of Open Addressing

- Operation time is length of probe sequence
- How long is it?
- In general, hard to answer.
- Introducing…
- "Uniform Hashing Assumption" (UHA):
  - Probe sequence is a uniform random permutation of [1..m]
  - (N.B. this is different to the simple uniform hashing assumption (SUHA))

Analysis under UHA

- Suppose:
  - a size-m table contains n items
  - we are using open addressing
  - we are about to insert new item
- Probability first prob successful?

$$\frac{m-n}{m} := p$$

Why? From UHA, probe sequence random permutation
Hence, first position probed random
m-n out of the m slots are unoccupied

Analysis (II)

- If first probe unsuccessful, probability second prob successful?

$$\frac{m-n}{m-1} \geq \frac{m-n}{m} = p$$

Why?

- From UHA, probe sequence random permutation
- Hence, first probed slot is random; the second probed slot is random among the remaining slots, etc.
- Since first probe unsuccessful, it probed an occupied slot
- Hence, the second probe is choosing uniformly from m-1 slots, among which m-n are still clean

Analysis (II)

- If first two probes unsuccessful, probability third prob successful?

$$\frac{m-n}{m-2} \geq \frac{m-n}{m} = p$$

- …

➔ every trial succeeds with probability ≥p

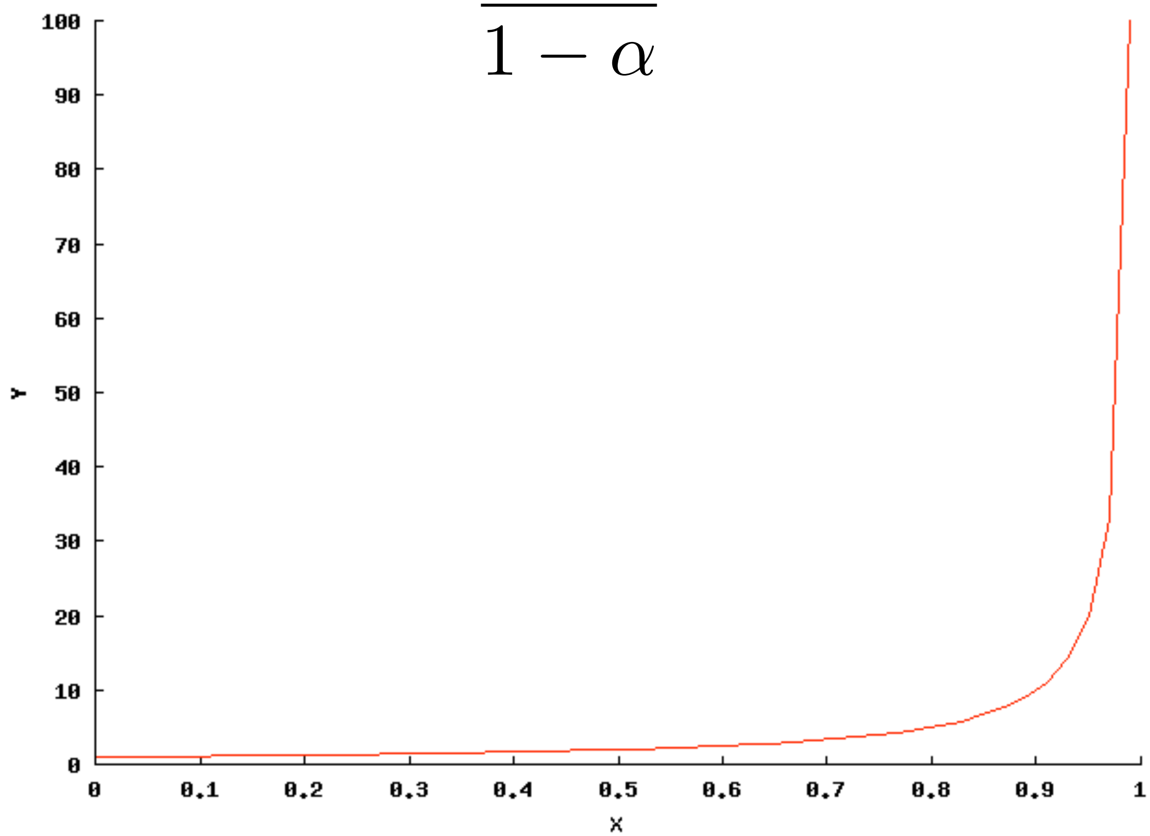
expected number of probes till success? $\leq \dfrac{1}{p} = \dfrac{1}{1-\alpha}$

e.g. if α=90%, expected number of probes is at most 10

Open Addressing vs. Chaining

- Open addressing skips linked lists
  - Saves space (of list pointers)
  - Better locality of reference
    - Array concentrated in m space
    - So fewer main-memory accesses bring it to cache
    - Linked list can wander all of memory
- Open addressing sensitive to α
  - As $\alpha \rightarrow 1$, access time shoots up

$$\frac{1}{1-\alpha}$$



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- Open addressing sensitive to α
  - As $\alpha \rightarrow 1$, access time shoots up
  - Cannot allow $\alpha > 1$
- Open addressing needs good hash to avoid clustering

ADVANCED HASHING

covered in recitation (for those who care)

Universal Hashing

- Get rid of simple uniform hashing assumption
- Create a family of hash functions
- When you start, pick one at random
- Unless you are unlucky, few collisions
  - Adversary doesn’t know what hash you will use
  - So cannot pick keys that collide in it

Universal Hash Family…

- …is a family (set) of hash functions such that, for any keys x and y, if you choose a random h from the family, $\Pr[h(x)=h(y)] = 1/m$
- Such a family produces few expected collisions
  - E[collisions with x] = E[number of y s.t. $h(x)=h(y)$]

$$= E\left[\Sigma_y \, 1_{h(x)=h(y)}\right]$$

$$= \Sigma_y \, E\left[1_{h(x)=h(y)}\right] \text{ (linearity of E)}$$

$$= \Sigma_y \Pr[\, h(x)=h(y) \,]$$

$$= n/m$$

Universal Families Exist!

- Suppose m is a prime p
- Define $h_{ab}(x) = a \cdot x + b \pmod{p}$
- If a and b are random elements in $\{0,\ldots,p-1\}$, then $h_{ab}(x)$ is a universal family
  - mod p is field, so you can divide/substract as well
  - Pick two keys x and y. What is the probability (over the choice of a, b) that the hashes of x and y collide?
  - It has to be that $a \cdot x + b = q \pmod{p}$ and $a \cdot y + b = q \pmod{p}$, for some q in $\{0,\ldots,p-1\}$
  - This is a linear system in a, b
    - Two variables, two equations
  - Unique solution---unique h_{ab} makes this happen
    - Probability of choosing this h_{ab} is $1/p^2$
  - Collide if $h_{ab}(x) = h_{ab}(y) = q$ for some q
  - hence overall probability of collision: $p/p^2 = 1/p = 1/m$
- Justifies multiplication hash

Even Better

- Perfect Hashing
  - Hash table with zero collisions
  - So don’t need linked lists
- Can’t guarantee for arbitrary keys
- But if you know keys in advance, can quickly find a hash function that works
  - E.g. for a fixed dictionary

Summary

- Hashing maps a large universe to a small range
- But avoids collisions
- Result:
  - Fast dictionary data structure
  - Fingerprints to save comparison time

- Next week: sorting