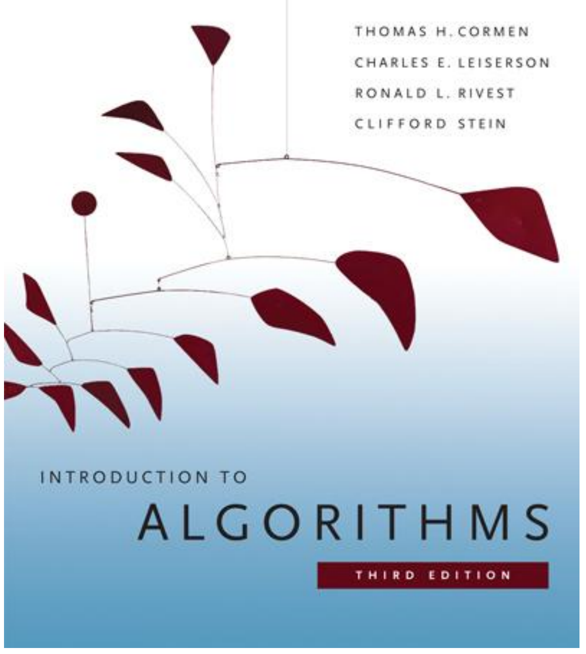
6.006- *Introduction to Algorithms*



Lecture 6

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CLRS: Chapter 17 and 32.2.

LAST TIME…

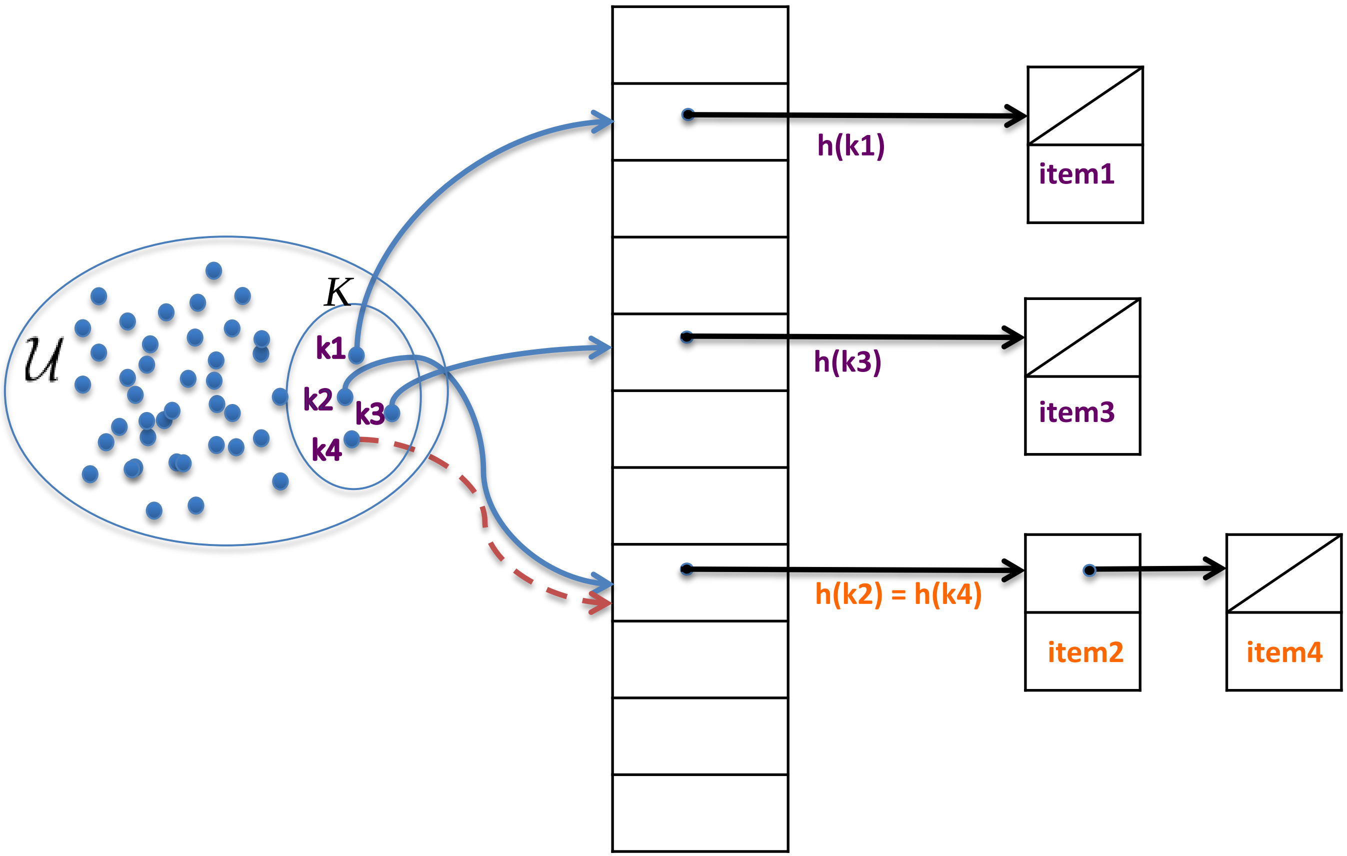
Dictionaries, Hash Tables

- **Dictionary**: Insert, Delete, Find a key
  - can associate a whole item with each key
- **Hash table**
  - implements a dictionary, by spreading items over an array
  - uses *hash function*

    h: Universe of keys (huge) → Buckets (small)
  - *Collisions*: Multiple items may fall in same bucket
  - *Chaining Solution*: Place colliding items in linked list, then scan to search
- **Simple Uniform Hashing** Assumption (SUHA):

  h is "random", uniform on buckets
  - Hashing n items into m buckets → expected "load" per bucket: n/m
  - If chaining used, expected search time O(1 + n/m)

Hash Table with Chaining



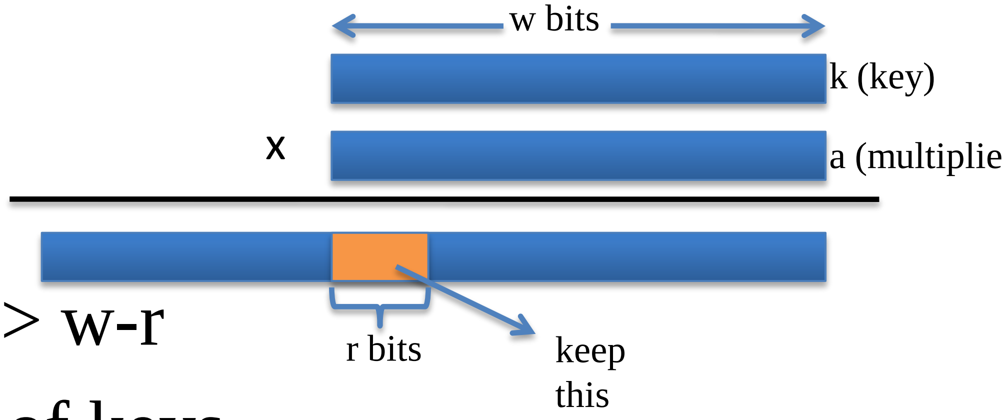
$\mathcal{U}$: *universe of all possible keys-huge set*

K : *actual keys-small set, but not known when designing data structure*

Hash Functions?

- Division hash
  - $h(k) = k \bmod m$
  - Fast if m is a power of 2, slow otherwise
  - Bad if e.g. keys are regular

- Multiplication hash
  - a an odd integer
  - $h(k) = (a \cdot k \bmod 2^w) >> w\text{-}r$
  - Better on regular sets of keys



Non-numbers?

- What if we want to hash e.g. strings?
- Any data is bits, and bits are a number
- E.g., strings:
  - Letters a..z can be "digits" base 26.
  - "the" $= t \cdot (26)^2 + h \cdot (26) + e$

    $= 19 \cdot (676) + 8 \cdot (26) + 5$

    $= 334157$
- Note: hash time is length of string, not O(1) (wait a few slides)

Longest Common Substring

- Strings S,T of length n, want to find longest common substring
- Algorithms from last time:

  $$O(n^4) \rightarrow O(n^3 \log n) \rightarrow O(n^2 \log n)$$

- Winner algorithm used a hash table of size n:

Binary search on maximum match length L; to check if a length works:

- Insert all length-L substrings of S in hash table
- For each length-L substring x of T
  - Look in bucket h(x) to see if x is in S

Runtime Analysis

- Binary search cost: $O(\log n)$ length values L tested
- For each length value L, here are the costly operations:
  - Inserting all L-length substrings of S: n-L hashes
    - Each hash takes L time, so total work $\Theta((n-L)L)=\Theta(n^2)$
  - Hashing all L-length substrings of T: n-L hashes
    - another $\Theta(n^2)$
  - Time for comparing substrings of T to substrings of S:
    - How many comparisons?
    - Under SUHA, each substring of T is compared to an expected $O(1)$ of substrings of S found in its bucket
    - Each comparison takes $O(L)$
    - Hence, time for all comparisons: $\Theta(nL)=\Theta(n^2)$
- So $\Theta(n^2)$ work for each length
- Hence $\Theta(n^2 \log n)$ including binary search

Faster?

- Amdahl's law: if one part of the code takes 20% of the time, then no matter how much you improve it, you only get 20% speedup
- Corollary: must improve all asymptotically worst parts to change asymptotic runtime
- In our case
  - Must compute sequence of n hashes faster
  - Must reduce cost of comparing in bucket

FASTER COMPARISON

Faster Comparison

- **First Idea:** when we find a match for some length, we can stop and go to the next value of length in our binary search.
- **But,** the real problem is "false positives"
  - Strings in same bucket that don't match, but we waste time on
- **Analysis:**
  - n substrings to size-n table: average load **1**
  - SUHA: for every substring x of T, there is **1** other string in x's bucket (in expectation)
  - Comparison work: **L** per string (in expectation)
  - So total work for all strings of T: $\mathbf{nL = \Theta(n^2)}$

Solution: Bigger table!

- What size?
- Table size **$m = n^2$**
  - n substrings to size-m table: average load **$1/n$**
  - SUHA: for every substring x of T, there is **$1/n$** other strings in x's bucket (in expectation)
  - Comparison work: **L/n** per string (in expectation)
  - So total work for all strings of T: **$n(L/n) = L = O(n)$**

- Downside?
  - Bigger table
  - (n^2 isn't realistic for large n)

Signatures

- Note n^2 table isn't needed for fast lookup
  - Size n enough for that
  - n^2 is to reduce cost of false positive compares
- So don't bother making the n^2 table
  - Just compute for each string another hash value in the larger range $1..n^2$
  - Called a signature
  - If two signatures differ, strings differ
  - Pr[same sig for two different strings] = $1/n^2$
    - (simple uniform hashing)

Application

- Hash substrings to size n table
- But store a signature with each substring
  - Using a second hash function to $[1..n^2]$
- Check each T-string against its bucket
  - First check signature, if match then compare strings
  - Signature is a small number, so comparing them is O(1)

strictly speaking O(logn); but if $n^2 < 2^{32}$ the signature fits inside a word of the computer; in this case, the comparison takes O(1)

Application

- Runtime Analysis:
  - for each T-string:

    O(bucket size)=O(1) work to compare signatures;
  - so overall O(n) time in signature comparisons
  - Time spent in string comparisons?

    L x (Expected Total Number of False-Signature Collisions)

    - n out of the n^2 values in $[1..n^2]$ are used by S-strings
    - so probability of a T-string signature-colliding with some S-string: n/n^2
    - hence total expected number of collisions 1

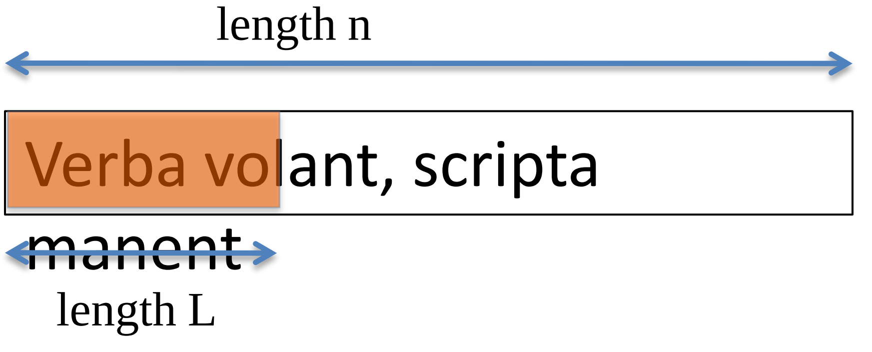
    so total time spent in String Comparisons is L

fine print: we didn't take into account the time needed to compute signatures; we can compute all signatures in O(n) time using trick described next…

FASTER HASHING

Rolling Hash

- We make a sequence of n substring hashes
  - Substring lengths L
  - Total time $O(nL) = O(n^2)$
- Can we do better?
  - For our particular application, yes!



Rolling Hash Idea

- e.g. hash all 3-substrings of "there"
- Recall division hash: x mod m
- Recall string to number:
  - First substring "the" = $t \cdot (26)^2 + h \cdot (26) + e$
- If we have "the", can we compute "her"?

"her"

$$= h \cdot (26)^2 + e \cdot (26) + r$$

$$= 26 \cdot \left(h \cdot (26) + e \right) + r$$

$$= 26 \cdot \left(t \cdot (26)^2 + h \cdot (26) + e - t \cdot (26)^2 \right) + r$$

$$= 26 \cdot \left(\text{"the"} - t \cdot (26)^2 \right) + r$$

- i.e. subtract first letter's contribution to number, shift, and add last letter

General rule

- Strings = base-b numbers
- Current substring S[i … i+L-1]

$$S[i] \cdot b^{L-1} + S[i+1] \cdot b^{L-2} + S[i+2] \cdot b^{L-3} \ldots + S[i+L-1]$$

$$- S[i] \cdot b^{L-1}$$

$$S[i+1] \cdot b^{L-2} + S[i+2] \cdot b^{L-3} \ldots + S[i+L-1]$$

$$b$$

$$S[i+1] \cdot b^{L-1} + S[i+2] \cdot b^{L-2} \ldots + S[i+L-1] \cdot b$$

$$+ \quad S[i+L]$$

$$S[i+1] \cdot b^{L-1} + S[i+2] \cdot b^{L-2} \ldots + S[i+L-1] \cdot b + S[i+L]$$

$$= S[i+1 \ldots i+L]$$

Mod Magic 1

- So: **S[i+1 … i+L] = b S[i … i+L-1]– b^L S[i] + S[i+L]**
- where

  S[i … i+L-1] = S[i] · b^{L-1} + S[i+1] · b^{L-2} + … + S[i+L-1] (*)
- **But** S[i … i+L-1] may be a huge number (so huge that we may not even be able to store in the computer, e.g. L=50, b=26)
- **Solution** only keep its ***division hash***: S[…] mod m
- This can be computed without computing S[…], using mod magic!
- Recall: **(ab) mod m = (a mod m) (b mod m) (mod m)**

  (a+b) mod m = (a mod m) + (b mod m) (mod m)

- With a clever parenthesization of (*): O(L) to hash string!

Mod Magic 2

- Recall: $S[i+1 \ldots i+L] = b\, S[i \ldots i+L-1] - b^L S[i] + S[i+L]$
- Say we have hash of $S[i \ldots i+L-1]$, can we still compute hash of $S[i+1 \ldots i+L]$?
- Still **mod magic** to the rescue!
- Job done in O(1) operations, if we know $b^L \bmod m$

→ Computing n-L hashes costs O(n)

O(L) time for the first hash

+O(L) to compute $b^L \bmod m$

+ O(1) for each additional hash

Summary

- Reduced compare cost to O(n)/length
  - By using a big hash table
  - Or signatures in a small table
- Reduced hash computation to O(n)/length
  - Rolling hash function
- Total cost of phases: O(n log n)

- Not the end: suffix tree achieves O(n)