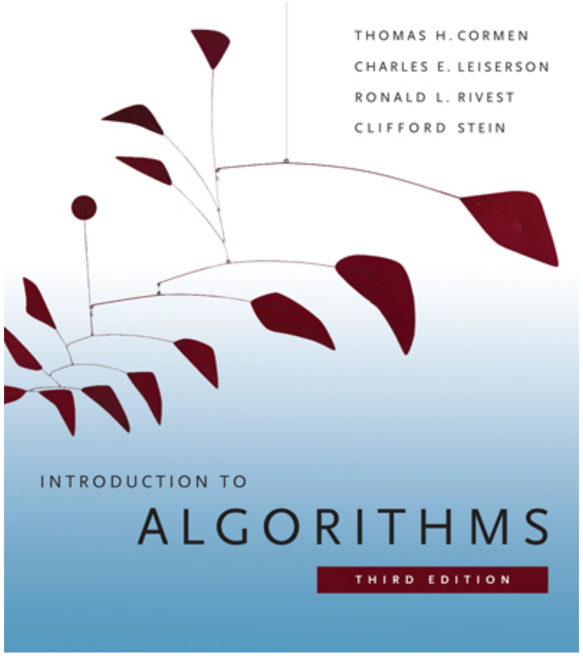
6.006- *Introduction to Algorithms*



Lecture 20

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Lecture overview

Dynamic Programming III

- review: longest common subsequence (LCS)
- recursion + memoization v.s. bottom up
  - (illustration with LCS)
- use of parent pointers
  - (illustration with LCS)
- knapsack problem
- text justification

Longest Common Subsequence (LCS)

- given two sequences x[1..m] and y[1..n], find a longest subsequence LCS(x,y) common to both:

x: A B C B D A B

y: B D C A B A

- denote the length of a sequence *s* by |*s*|
- first get |LCS(x,y)|

LCS: A recurrence

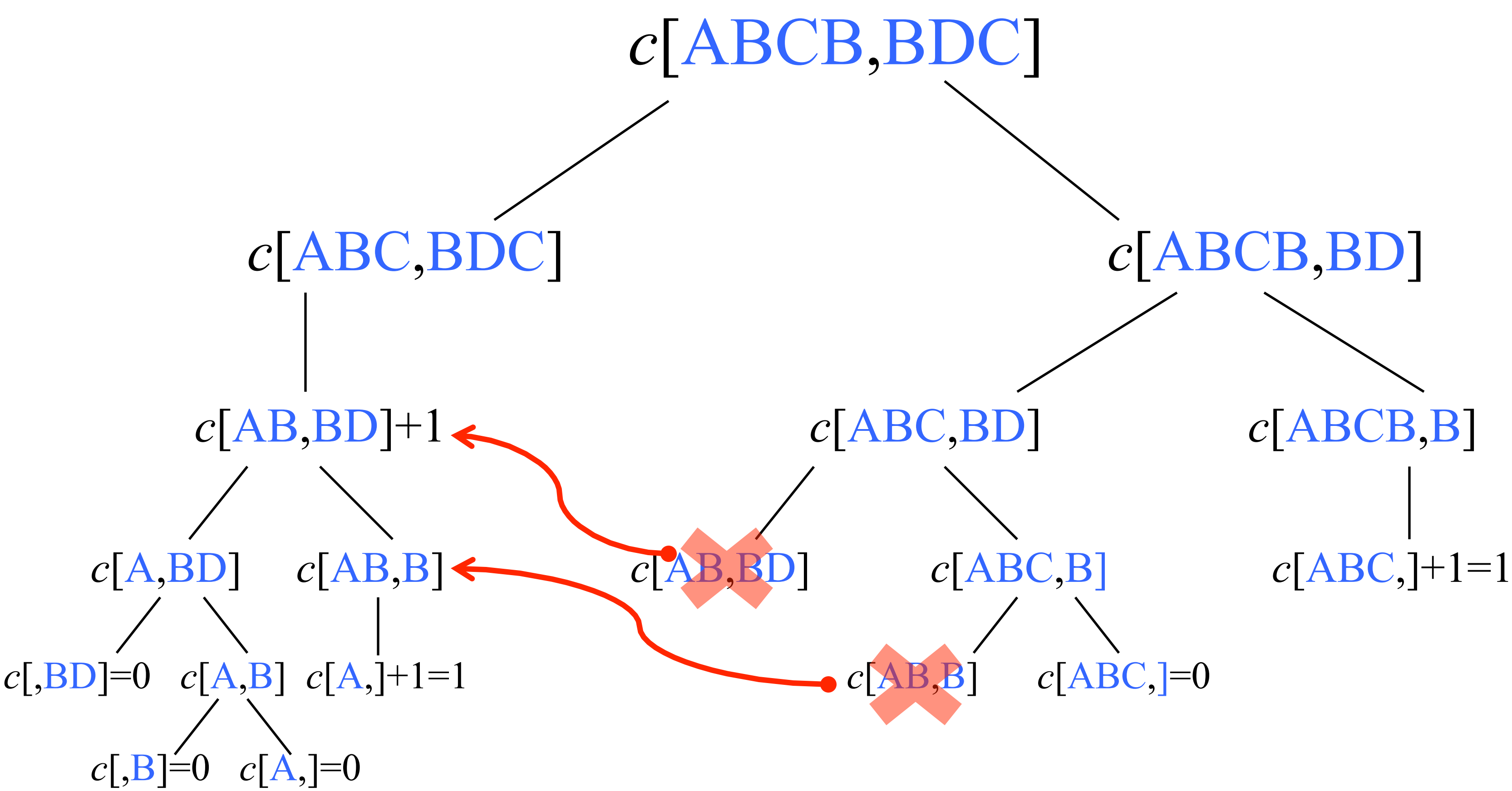
- consider prefixes of x and y
  - x[1..i] ith prefix of x[1..m]
  - y[1..j] jth prefix of y[1..n]
- define c[i,j] = |LCS(x[1..i],y[1..j])

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0, \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j, \\ \max(c[i-1, j], c[i, j-1]) & \text{if } i, j > 0 \text{ and } x_i \neq y_j. \end{cases}$$

running time is .... $O(n \times m)$.... (if done well !)

LCS recursion+memoization

$$c[\alpha,\beta] = \begin{cases} 0 & \text{if } \alpha \text{ empty or } \beta \text{ empty,} \\ c[\mathit{prefix}\alpha, \mathit{prefix}\beta] + 1 & \text{if } \mathrm{end}(\alpha) = \mathrm{end}(\beta), \\ \max(c[\mathit{prefix}\alpha,\beta], c[\alpha,\mathit{prefix}\beta]) & \text{if } \mathrm{end}(\alpha) \neq \mathrm{end}(\beta). \end{cases}$$



LCS – bottom up & pointers

|LCS(x, y)|
$m \leftarrow length[x]$
$n \leftarrow length[y]$
for $i \leftarrow 1$ **to** m
    do $c[i, 0] \leftarrow 0$
for $j \leftarrow 0$ **to** n
    do $c[0, j] \leftarrow 0$
for $i \leftarrow 1$ **to** m
    do for $j \leftarrow 1$ **to** n
        do if $x_i = y_j$
            then $c[i, j] \leftarrow c[i\text{-}1, j\text{-}1] + 1$
                $p[i, j] \leftarrow$ "↖"
            else if $c[i\text{-}1, j] \geq c[i, j\text{-}1]$
                then $c[i, j] \leftarrow c[i\text{-}1, j]$
                    $p[i, j] \leftarrow$ "↑"
                else $c[i, j] \leftarrow c[i, j\text{-}1]$
                    $p[i, j] \leftarrow$ "←"
return c and p

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0, \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j, \\ \max(c[i-1, j], c[i, j-1]) & \text{if } i, j > 0 \text{ and } x_i \neq y_j. \end{cases}$$

Example

x:	A	B	C	B
y:	B	D	C	

	y_j	B	D	C
x_i	0	0	0	0
A	0	↑ 0	↑ 0	↑ 0
B	0	↖ 1	← 1	← 1
C	0	↑ 1	↑ 1	↖ 2
B	0	↖ 1	↑ 1	↑ 2

Use of parent pointers

- we found length of LCS, what about actual LCS?
- using the "parent pointers" p
  - p remembers if c[i,j] used c[i-1, j-1], c[i, j-1], or c[i-1,j]
  - starting at c[m,n]:
    - if c[m-1,n-1], then x[m]=y[n] is part of *opt*
      - put it at end and output *opt* from c[m-1,n-1]
    - else, output *opt* from c[m-1,n] or c[m,n-1]

Constructing an LCS

```
PRINT-LCS (p, x, i, j)
if i = 0 or j = 0
    then return
if p[i, j] = "↖"
    then PRINT-LCS(p, x, i-1, j-1)
            print x_i
    elseif p[i, j] = "↑"
            then PRINT-LCS(p, x, i-1, j)
else PRINT-LCS(p, x, i, j-1)
```

initial call is PRINT-LCS (p, x, m, n)

running time: $O(m+n)$

Example

x: A B C B

y: B D C

	y_j	B	D	C
x_i	0	0	0	0
A	0	↑ 0	↑ 0	↑ 0
B	0	↖ 1	← 1	← 1
C	0	↑ 1	↑ 1	↖ 2
B	0	↖ 1	↑ 1	↑ 2

Bottom-Up DP

- we've been looking at DP recurrences
  - which suggests recursive implementations
  - and memoize results as you get them
- can also solve "bottom up"
  - compute sub-problems before super-problem
  - put results in memo table for later use
- how to order problems to ensure this works?

The DP DAG

- define a graph representing DP
  - sub-problems are vertices
  - edge $x \rightarrow y$ if problem x depends on problem y
- what order of problem solving works?
  - need order where x follows y if $x \rightarrow y$
  - Topological Sort!
  - can do so if graph is a DAG
  - what if not?
    - cyclic problem dependency
    - can't use DP

Knapsack Problem

- Knapsack (or cart) of size S
- Collection of n items; item i has size s_i and value v_i
- Goal: choose subset with $\Sigma_i \, s_i < S$ maximizing $\Sigma_i \, v_i$
- Ideas?
  - try all possible subsets: 2^n
  - greedy?
    - choose items maximizing value ?
    - choose items maximizing value/size
      - what if they don't exactly fit?

Some bad and better news

- For arbitrary (real) , Knapsack is hard (NP-hard)
  - no polynomial time algorithm in 30 years of trying
  - it's exactly as hard as several thousand other important problems
  - and we haven't been able to find polynomial time algorithms for them for 30 years of trying either
  - most folks think there is none
- Better news:
  - There is a DP algorithm if sizes are integers

First attempt

- subproblem?
  - Val[i] = Best value obtained for items[i:n]
- guess?
  - whether or not to include item i
- recurrence?
  - Val[i] = Val[i+1]
    or v_i + Val[i+1] if total size < S?
- not a well-defined recurrence: doesn’t have enough info to tell if item i will fit

Second Attempt

- Solve a more complicated problem
  - initial problem is a special case
  - the complicated version has a recursion
- Val[i,X] = max value for items[i:n] if space is X
- Recurrence:
  - if $s_i >$ X then don't include i, otherwise decide with
  - $Val[i, X] = \max(Val[i + 1, X], v_i + Val[i + 1, X - s_i])$
  - Opt = Val[0,S]

Analysis

- Is the recurrence a DAG?
  - yes, each problem depends on bigger i and smaller X
  - compute by decreasing i and increasing X
- Runtime?
  - each subproblem has 2 guesses: O(1)
  - one subproblem for each i, $X<S$
  - O(nS) subproblems
  - Total time: O(nS)
- Is this polynomial?

Text Justification – Word Processing

- A user writes stream of text
- WP has to break it into lines that aren't too long
- obvious algorithm => greedy:
  - put as much on first line as possible
  - then continue to lay out rest
  - used by MSWord, OpenOffice
- Problem: suboptimal layouts !!

A Better Approach

- define an objective function
  - measure of how good a given layout is
  - not an algorithm, just a metric
- optimize the objective
  - here's where you think of algorithm

Layout Function

- want to penalize big spaces
- what objective would do that?
  – sum of leftover spaces?
  – that’s constant for a given number of lines (just total space minus number of characters)
- should penalize big spaces “extra”
  – (LaTeX uses sum of cubes of leftovers)

Formalize

- input: array of words (lengths) w[0..n]
- split into lines L_1, L_2 …
- badness(L) = (page width – total length(L))^3
  - (or ∞ if total length > page width)
- objective: break into lines L_1, L_2… minimizing
  Σ_i badness(L_i)

Can We DP?

- Subproblems?
  - DP[i] = min badness for words w[i:n]
  - n subproblems where n is number of words
- Guesses for problem i?
  - Where to end first line in optimal layout
- Recurrence?
  - DP[i] = min badness(i,j) + DP[j] for j in range(i+1,n)
  - DP[n]=0
  - OPT = DP[0]
- Runtime? $O(n^2)$?