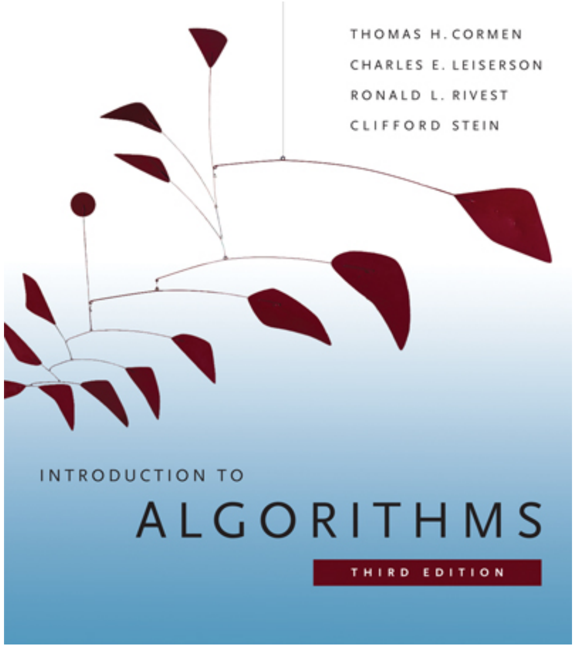
6.006- *Introduction to Algorithms*



Lecture 12

Prof. Constantinos Daskalakis

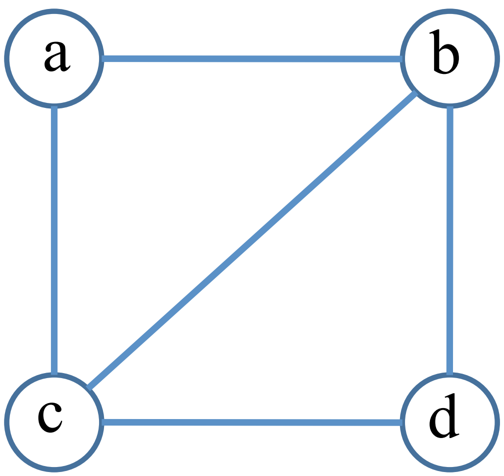
CLRS 22.2-22.3

Graphs

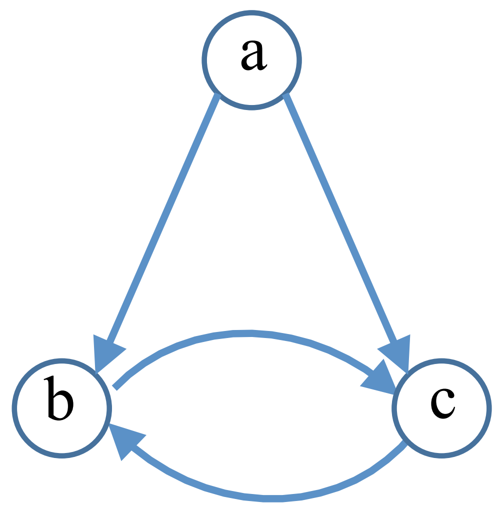
- $G=(V,E)$
- V a set of vertices
  - Usually number denoted by n
- $E \subseteq V \times V$ a set of edges (pairs of vertices)
  - Usually number denoted by m
  - Note $m \leq n(n-1) = O(n^2)$
- Flavors:
  - Pay attention to order of vertices in edge: *directed* graph
  - Ignore order: *undirected* graph
    - Then only $n(n-1)/2$ possible edges

Examples

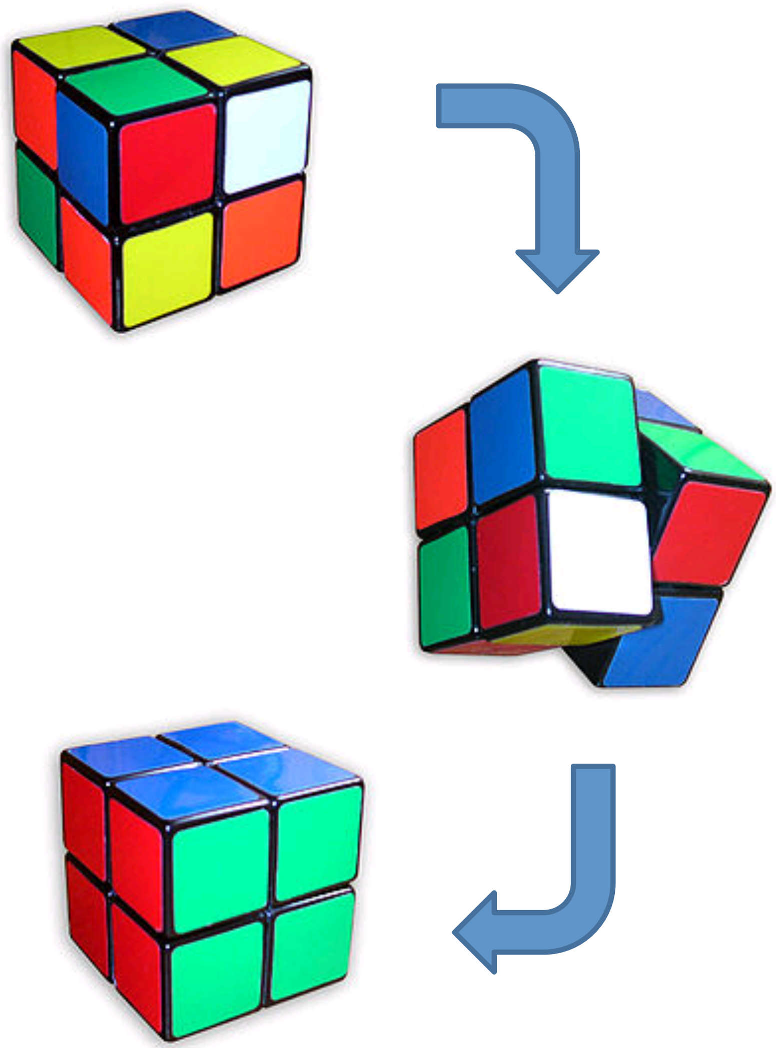
- ***Undirected***
- V={a,b,c,d}
- E={{a,b}, {a,c}, {b,c}, {b,d}, {c,d}}



- ***Directed***
- V = {a,b,c}
- E = {(a,c), (a,b) (b,c), (c,b)}



Pocket Cube

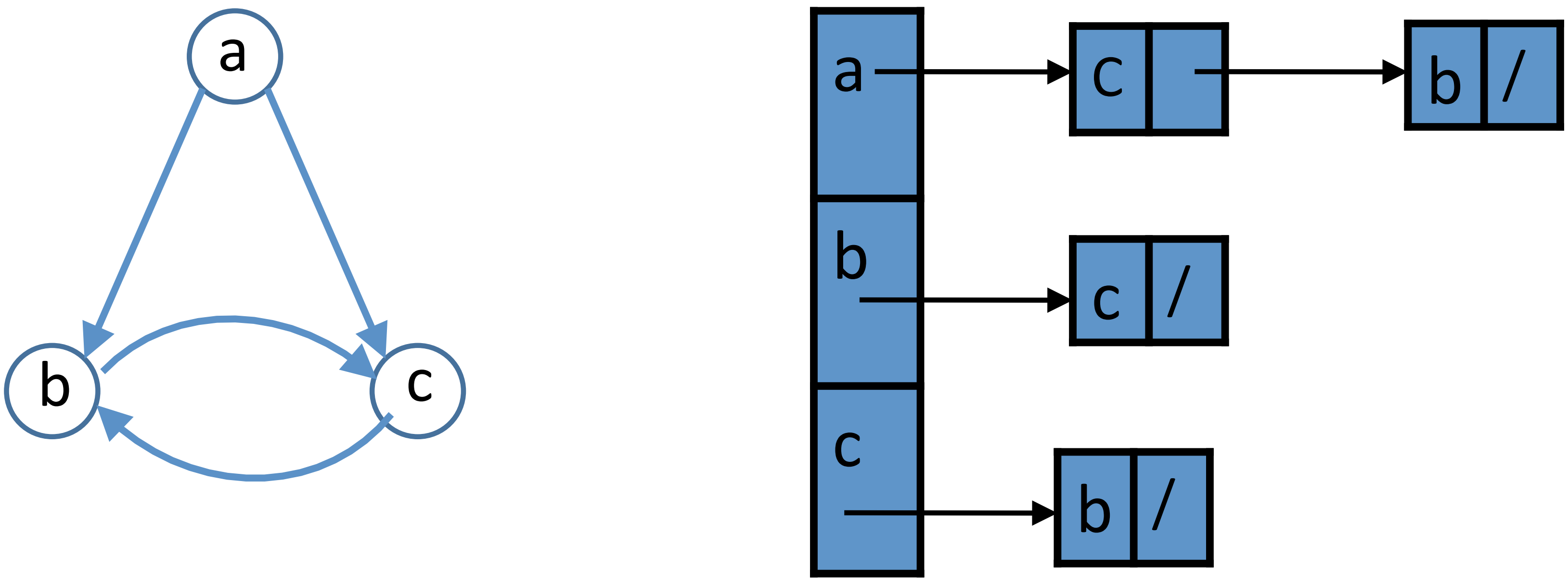


- $2 \times 2 \times 2$ Rubik's cube
- Configurations are adjacent, if one can be obtained from the other by quarter turns
- ***Basic Question:*** is solved state reachable with such moves from the starting state?

Representation

- To solve graph problems, must examine graph
- So need to represent in computer
- Four representations with pros/cons
  - *Adjacency lists* (of neighbors of each vertex)
  - *Incidence lists* (of edges from each vertex)
  - *Adjacency matrix* (of which pairs are adjacent)
  - *Implicit representation* (as neighbor function)

Example



Searching Graph

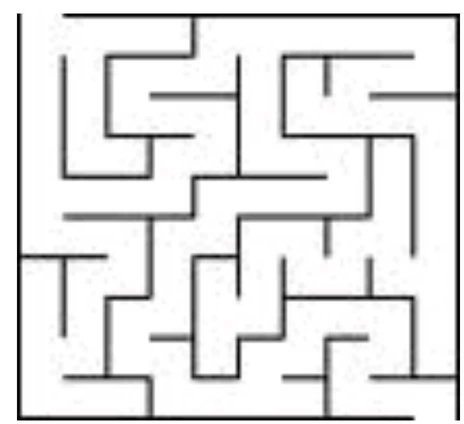
- We want to get from current Rubik state to "solved" state
- How do we explore?

Breadth First Search

- Start with vertex v
- List all its neighbors (distance 1)
- Then all their neighbors (distance 2)
- Etc.

Depth First Search

- Like exploring a maze
- From current vertex, move to another
- Until you get stuck
- Then backtrack till you find a new place to explore



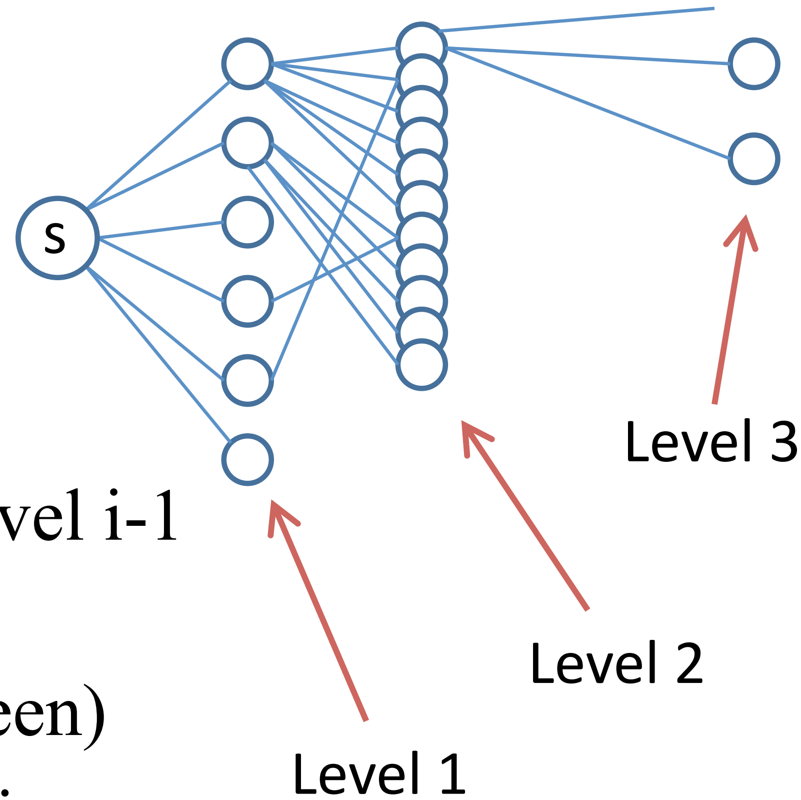
Problem: Cycles

- What happens if unknowingly revisit a vertex?
- BFS: get wrong notion of distance
- DFS: may get in circles
- Solution: mark vertices
  - BFS: if you’ve seen it before, ignore
  - DFS: if you’ve seen it before, back up

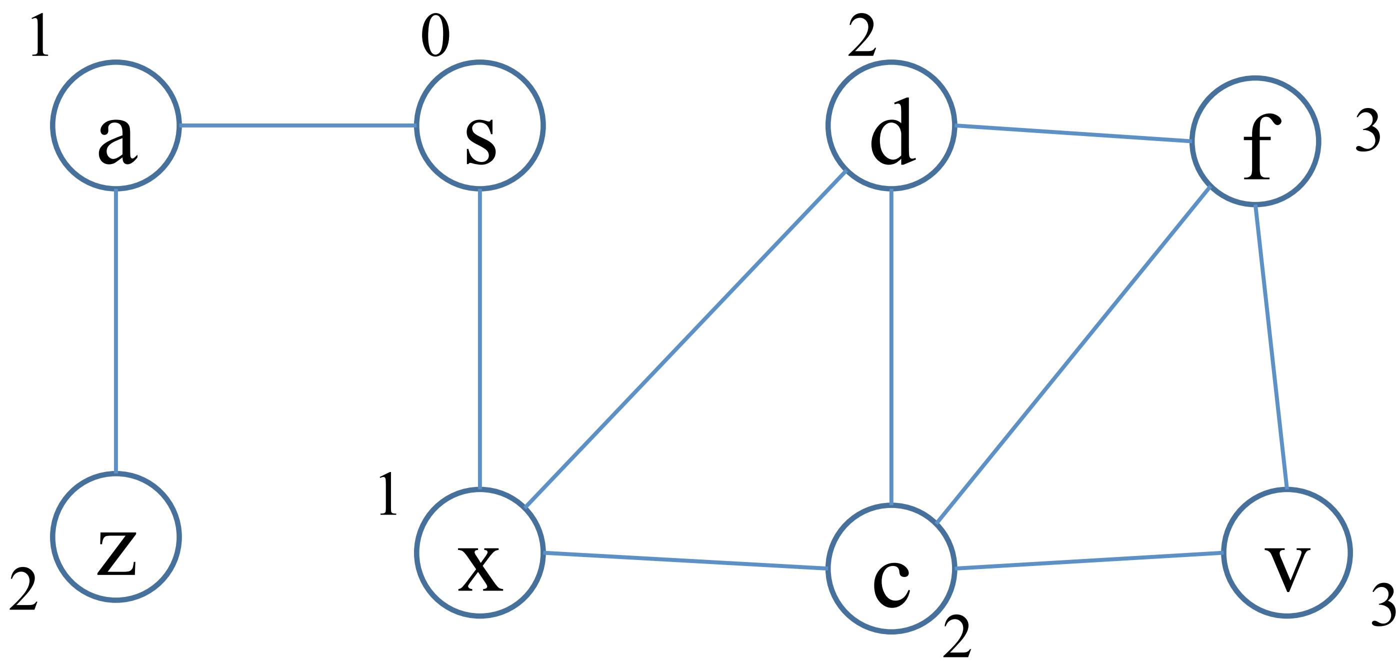
Breadth First Search (BFS)

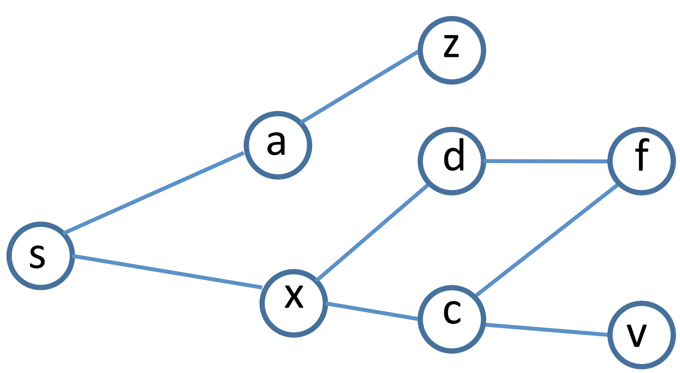
Outline

- Initial vertex s
  - Level 0
- For i=1,…
  grow level i
  - Find all neighbors of level i-1 vertices
  - (except those already seen)
  - i.e. level i contains vertices reachable via a path of i edges and no fewer



Example





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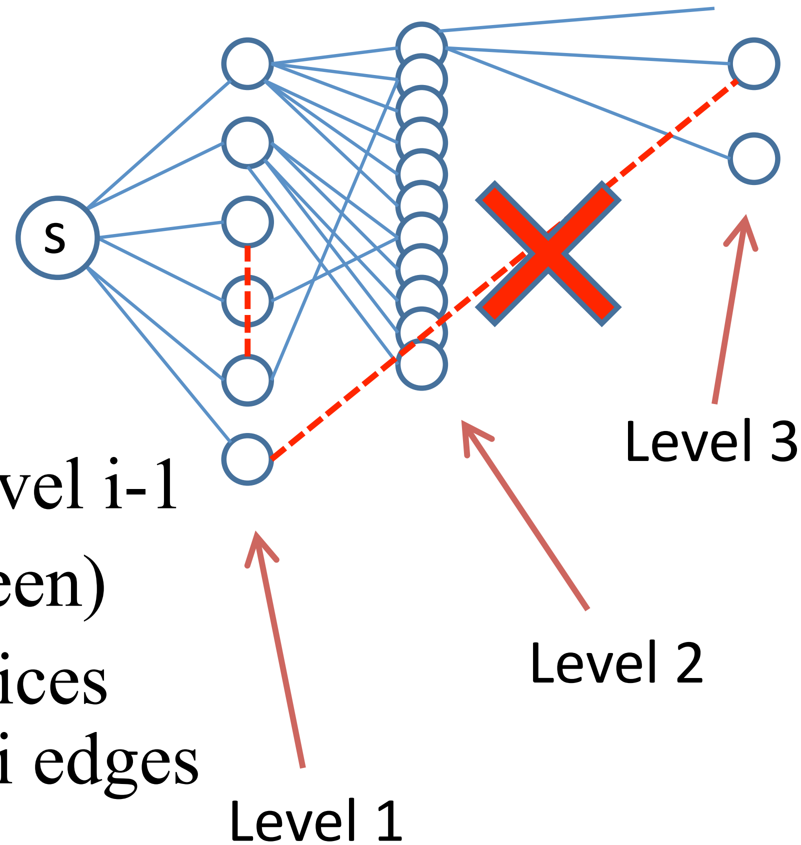
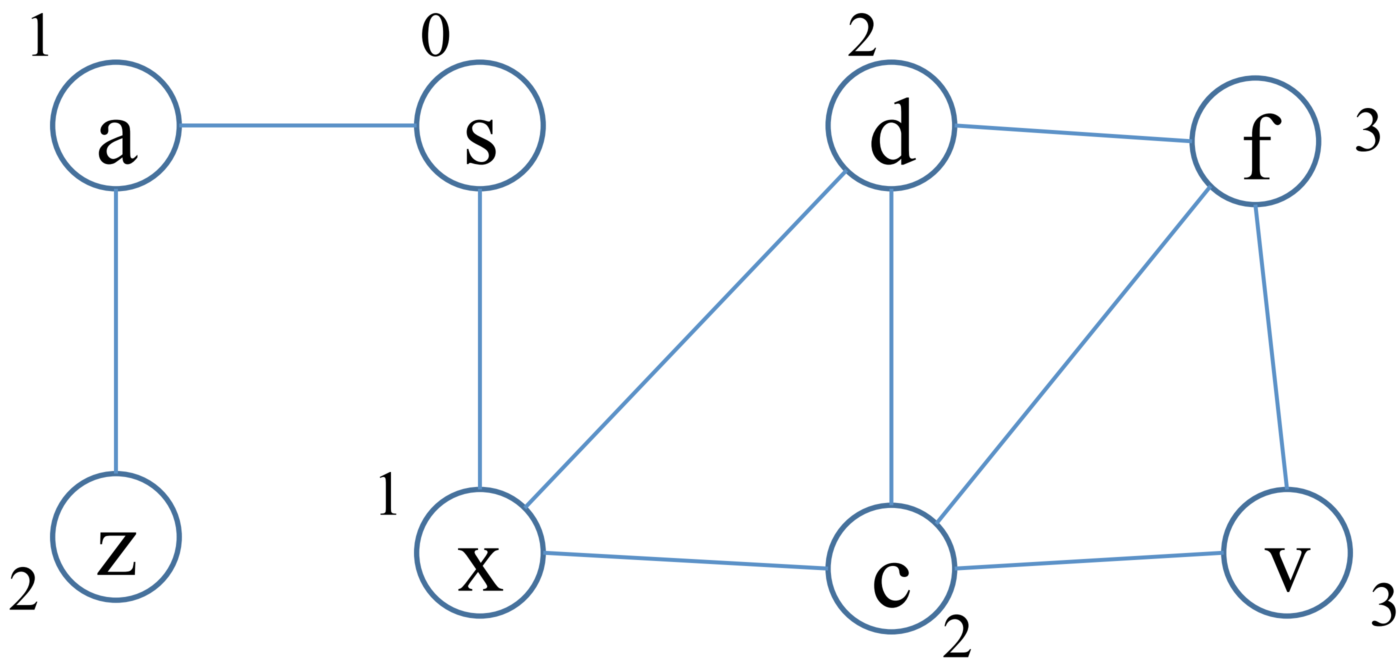
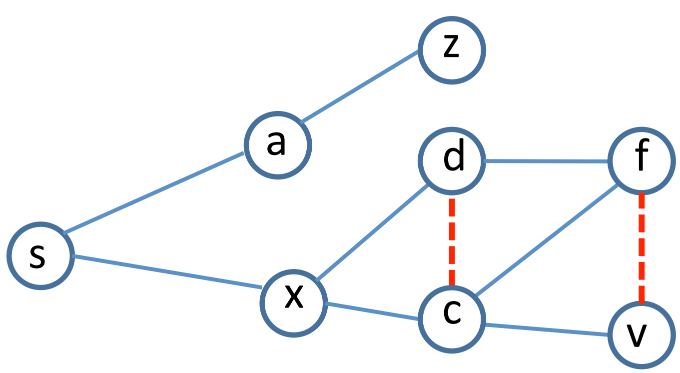


- Where can the other edges of the graph be?
  - Only between nodes in same or adjacent levels

Example





Algorithm

- BFS(V,Adj,s)

```
level={s: 0}; parent = {s: None}; i=1
frontier=[s]                          #previous level, i-1
while frontier
    next=[]                           #next level, i
    for u in frontier
        for v in Adj[u]
            if v not in level         #not yet seen
                level[v] = i          #level of u+1
                parent[v] = u
                next.append(v)
    frontier = next
    i += 1
```

Analysis: Runtime

- Vertex v appears at the ***frontier*** at most once
  - Since then it has a level
  - And nodes with a level aren't added again
  - Total time spent adding nodes to ***frontier*** O(n)
- Adj[v] only scanned once
  - Just when v is in ***frontier***
  - Total time $\sum_v |\mathrm{Adj}[v]|$
    - This sum counts each "outgoing" edge
    - So O(m) time spend scanning adjacency lists
- Total: O(m+n) time --- "Linear time"

Analysis: Correctness

i.e. why are all nodes reachable from s explored?

- **Claim**: If there is a path of L edges from s to v, then v is added to next when i=L or before
- **Proof**: induction
  - Base case: s is added before setting i=1
  - Path of length L from s to v
  - → path of length L-1 from s to u, and edge (u,v)
  - By induction, add u when i=L-1 or before
  - If v has not already been inserted in next before i=L, it gets added when scan u at i=L
  - So it happens when i=L or before

Shortest Paths

- From correctness analysis, conclude more:
  - Level[v] is length of shortest s—v path
- Parent pointers form a shortest paths tree
  - Which is union of shortest paths to all vertices
- To find shortest path, follow parent pointers
  - Will end up at s

Depth First Search (DFS)

Outline

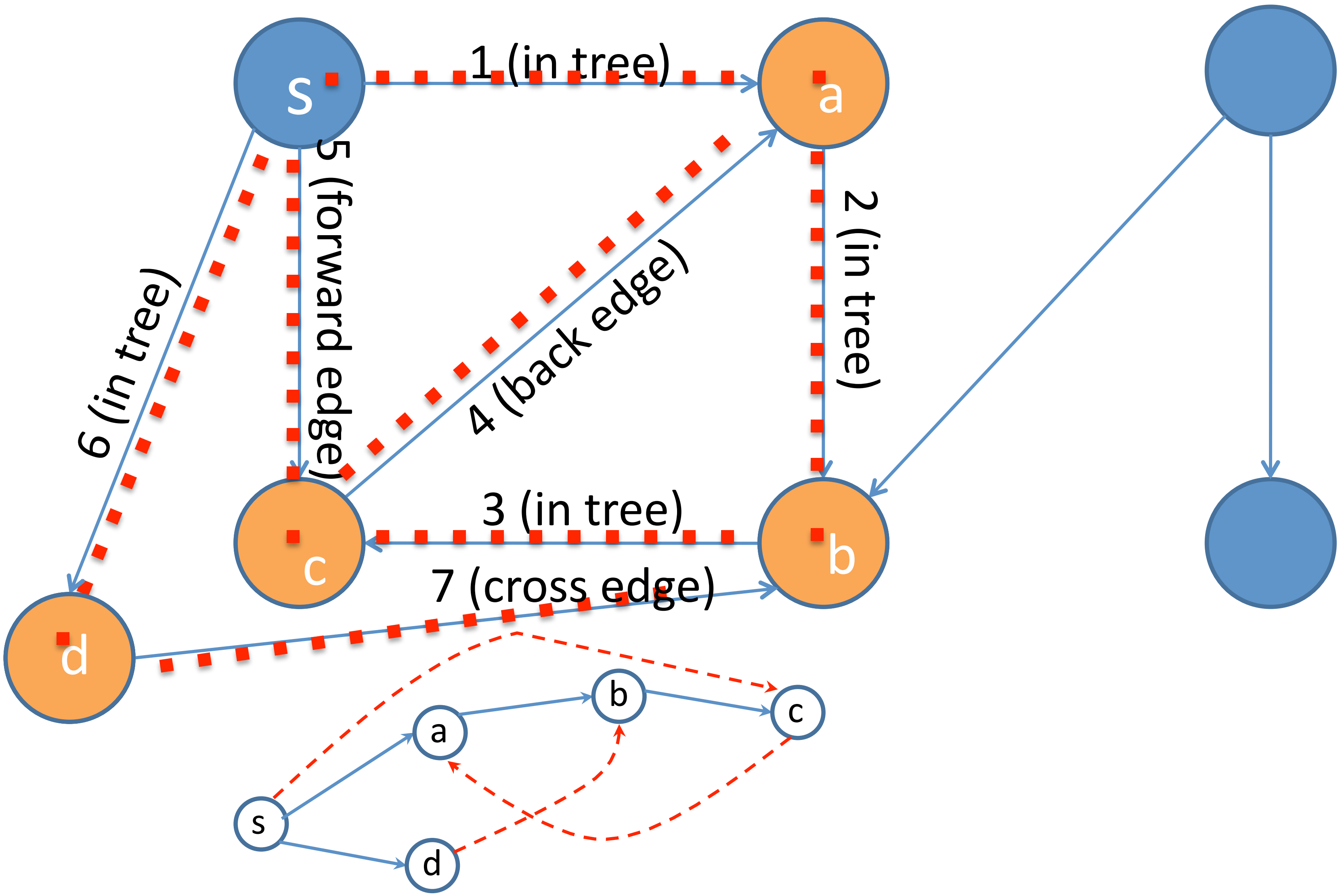
- Explore a maze
  - Follow path until you get stuck
  - Backtrack along breadcrumbs till find new exit
  - i.e. recursively explore

Algorithm

- ***parent*** = {s: None}
- call DFS-visit (V, Adj, s)

```
Routine DFS-visit (V, Adj, u)
    for v in Adj[u]
        if v not in parent              #not yet seen
            parent[v] = u
            DFS-visit (V, Adj, v)       #recurse!
```

Demo (from s)



s
a
b
c
d
1 (in tree)
2 (in tree)
3 (in tree)
4 (back edge)
5 (forward edge)
6 (in tree)
7 (cross edge)
s
a
b
c
d

Runtime Analysis

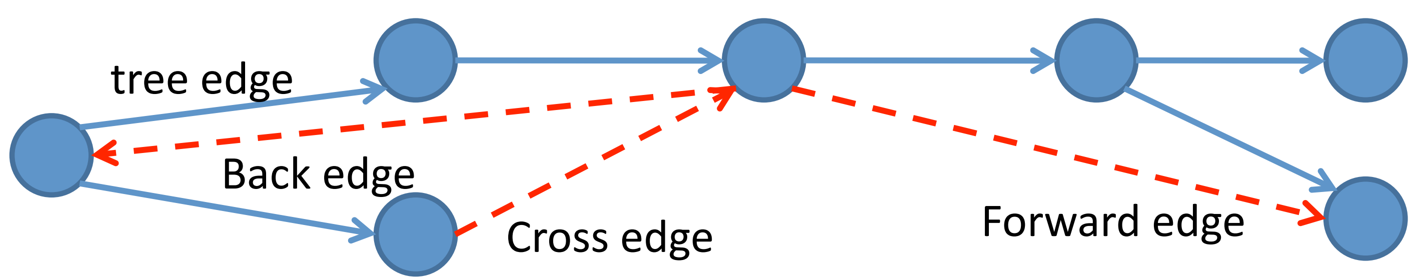
- Quite similar to BFS
- DFS-visit only called once per vertex v
  - Since next time v is in ***parent*** set
- Edge list of v scanned only once (in that call)
- So time in DFS-visit is 1/vertex + 1/edge
- So time is O(n+m)

Correctness?

- Trickier than BFS
- Can use induction on length of shortest path from starting vertex
  - Induction Hypothesis: “each vertex at distance k is visited”
  - Induction Step:
    - Suppose vertex v at distance k
    - Then some u at distance k-1 with edge (u,v)
    - u is visited (by induction hypothesis)
    - Every edge out of u is checked
    - If v wasn’t previously visited, it gets visited from u

Edge Classification

- Tree edge used to get to new child
- Back edge leads from node to ancestor in tree
- Forward edge leads to descendant in tree
- Cross edge leads to a different subtree
- To label what edge is of what type, keep global time counter and store interval during which vertex is on recursion stack



Tradeoffs

- Solving Rubik's cube?
  - BFS gives shortest solution
- Robot exploring a building?
  - Robot can trace out the exploration path
  - Just drops markers behind
- Only difference is "next vertex" choice
  - BFS uses a queue
  - DFS uses a stack (recursion)