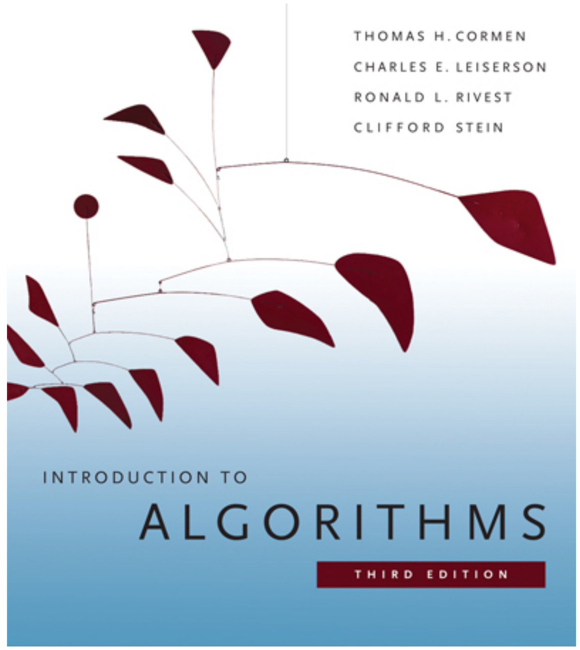
6.006- *Introduction to Algorithms*



Lecture 9

Prof. Constantinos Daskalakis

CLRS: 2.1, 2.2, 2.3, 6.1, 6.2, 6.3 and 6.4.

Lecture Overview

Priority Queues

Heaps

Heapsort

Priority Queue

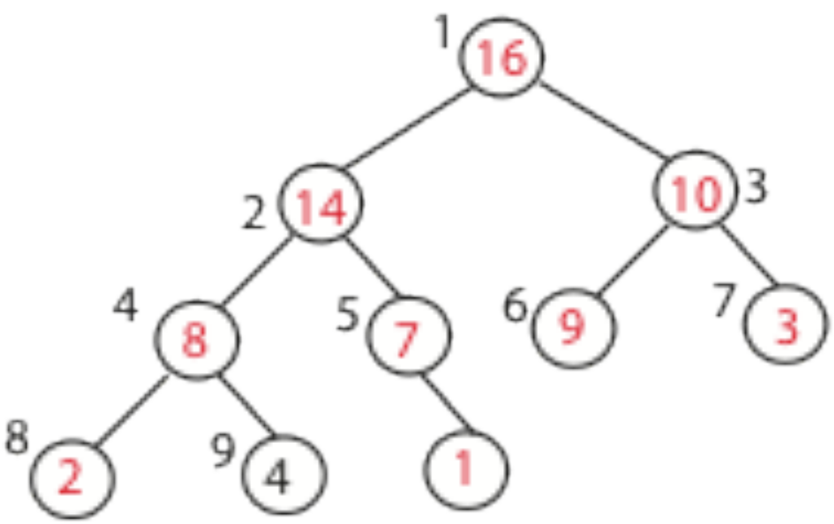
This is an ***abstract datatype*** implementing a set S of elements, each associated with a key, supporting the following operations:

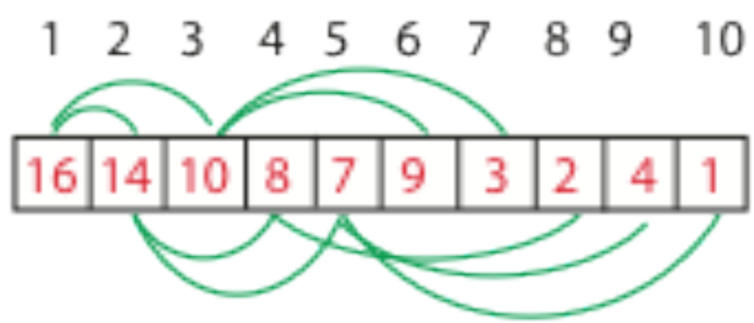
insert(S, x) :	insert element x into set S
max(S) :	return element of S with largest key
extract_max(S) :	return element of S with largest key and remove it from S
increase_key(S, x, k) :	change the key-value of element x to the value k (assumed to be as large as current value)

Heap

An implementation of a priority queue. It is an array object, visualized as a nearly complete binary tree.

Heap Property: The key of a node is $\geq$ than the keys of its children.





Visualizing an Array as a Tree

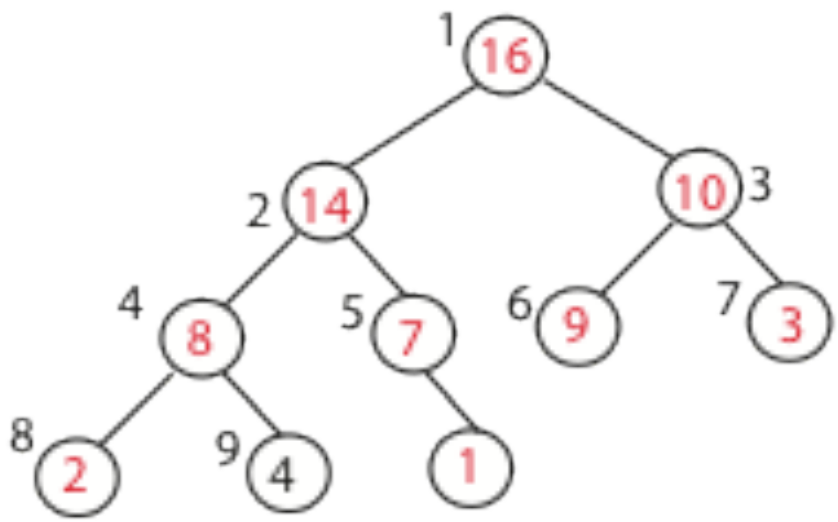
root of tree: first element in the array, corresponding to index = 1

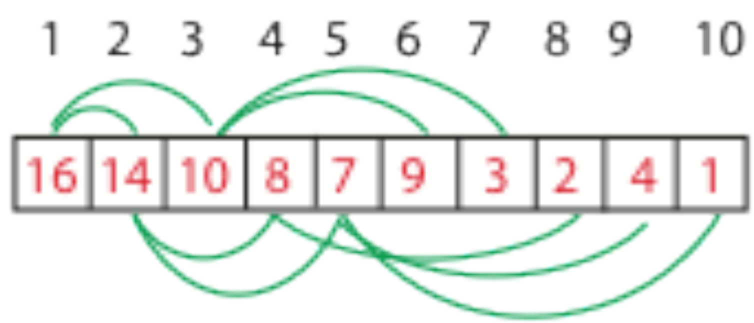
If a node's index is i then:

$$\text{parent}(i) = \left\lfloor \frac{i}{2} \right\rfloor$$; returns index of node's parent, e.g. parent(5)=2

$\text{left}(i) = 2i;$ returns index of node's left child, e.g. left(4)=8

$\text{right}(i) = 2i + 1;$ returns index of node's right child, e.g. right(4)=9





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$\mathrm{left}(i) = 2i;$ returns index of node's left child, e.g. left(4)=8

$\mathrm{right}(i) = 2i + 1;$ returns index of node's right child, e.g. right(4)=9

Note: no pointers required! Height of a binary heap $O(\log_2 n)$.

Heap-Size Variable

For flexibility we may only need to consider the first few elements of an array as part of the heap.

The variable heap-size denotes the number of items of the array that are part of the heap:

A[1],…, A[heap-size];

Max-Heaps vs Min-Heaps

Max Heaps satisfy the Max-Heap Property

$$\text{for all } i, \quad A[i] \geq \max\{\, A[\text{left}(i)]\, ,\, A[\text{right}(i)]\}$$

Min Heaps satisfy the Min-Heap Property

$$\text{for all } i, \quad A[i] \leq \min\{\, A[\text{left}(i)]\, ,\, A[\text{right}(i)]\}$$

Operations with Heaps

build_max_heap : produce a max-heap from an unordered array in O(n);

max_heapify : correct a single violation of the heap property occurring at the root of a subtree in O(log n);

insert, extract_max : O(log n)

heapsort : sort an array of size n in O(n log n) using heaps

Max_heapify

Max_heapify

correct a single violation of the heap property occurring at the root of a subtree in O(log n);

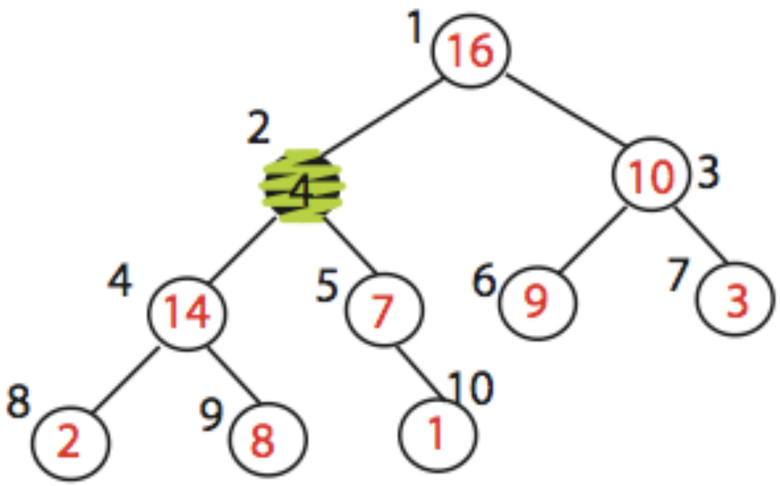
Assume that the trees rooted at left(i) and right(i) are max-heaps, but element A[i] violates the max-heap property;

i.e. A[i] is smaller than at least one of A[left(i)] or A[right(i)].

The goal is to correct the violation.

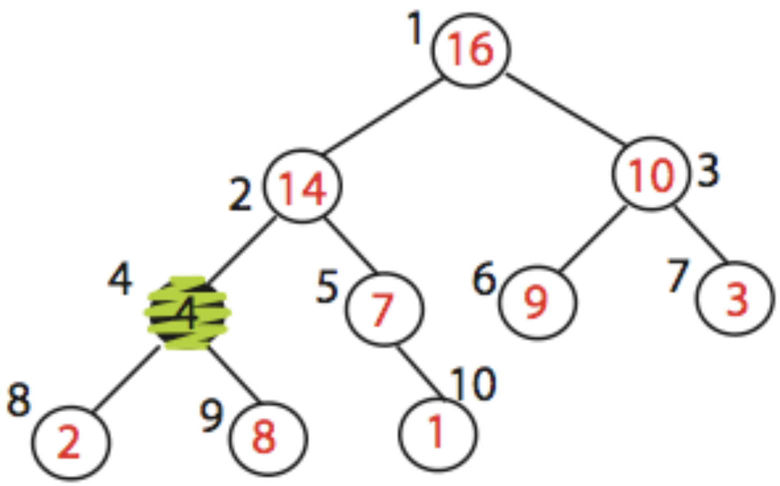
Do this by trickling element A[i] down the tree, making the subtree rooted at index i a max-heap.

Max_heapify (Example)



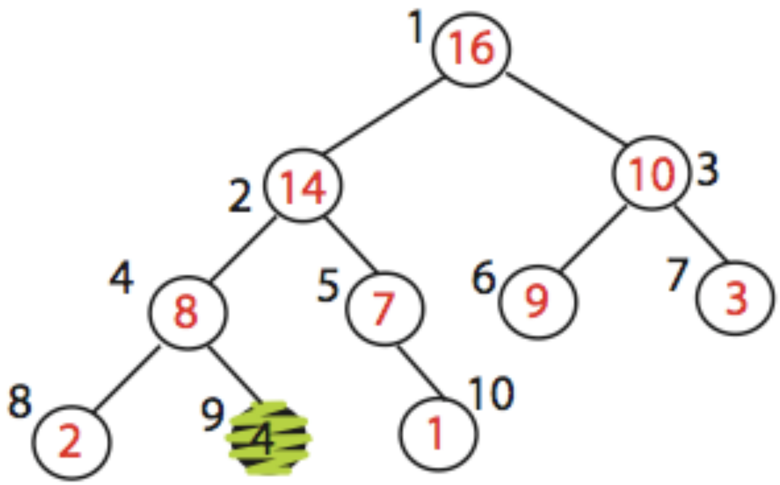
MAX_HEAPIFY (A,2)
heap_size[A] = 10

Max_heapify (Example)



Exchange A[2] with A[4]
Call MAX_HEAPIFY(A,4)
because max_heap property
is violated

Max_heapify (Example)



Exchange A[4] with A[9]
No more calls

Max_heapify (Pseudocode)

Max_heapify (A, *i*)

Find the index of the largest element among A[*i*], A[left(*i*)] *and* A[right(*i*)]

```
l ← left(i)
r ← right(i)
if l ≤ heap-size(A) and A[l] > A[i]
   then largest ← l
   else largest ← i
if r ≤ heap-size(A) and A[r] > A[largest]
   then largest ← r
```

If this index is different than i, exchange A[*i*] *with largest element; then recurse on subtree*

```
if largest ≠ i
   then exchange A[i] and A[largest]
        MAX_HEAPIFY(A, largest)
```

IMPORTANT NOTE: If element A[i] is smaller than both A[left(*i*)] *and* A[right(*i*)], I insist on swapping it with the largest of the two and not with either one of them, arbitrarily.

Build_Max_heap

Build_Max_Heap(A)

Convert A[1…n] to a max heap.

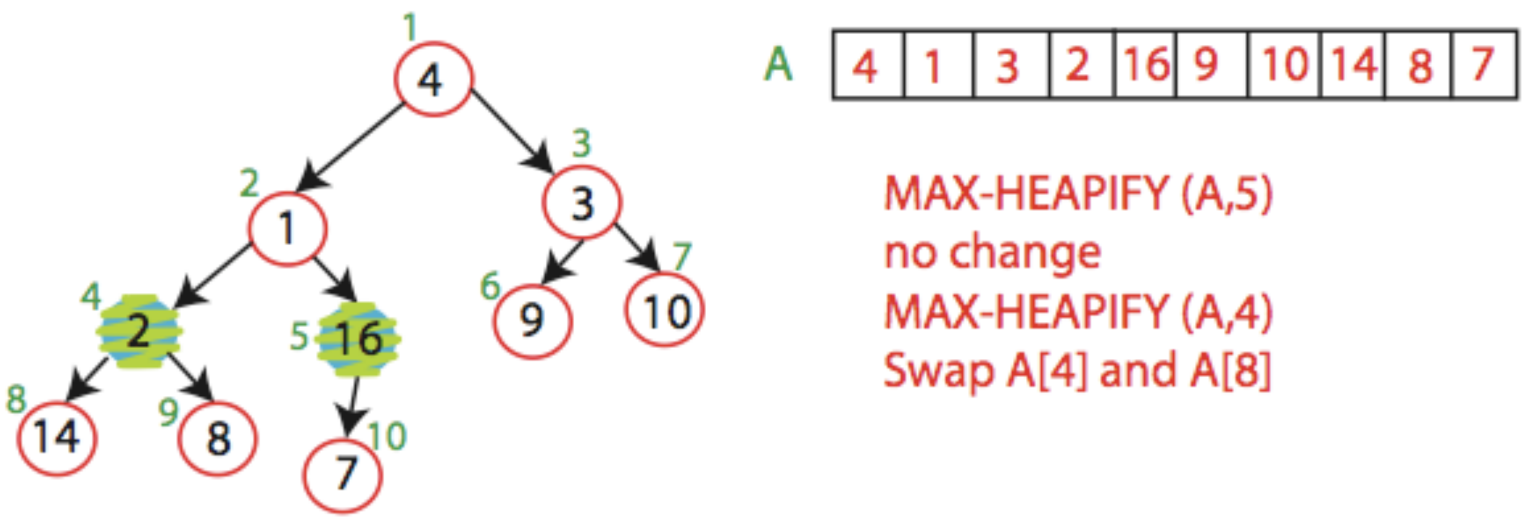
Observation: Elements $A[\lfloor n/2 \rfloor + 1 \ldots n]$ are leaves of the tree

because $2i > n$, for all $i \geq \lfloor n/2 \rfloor + 1$

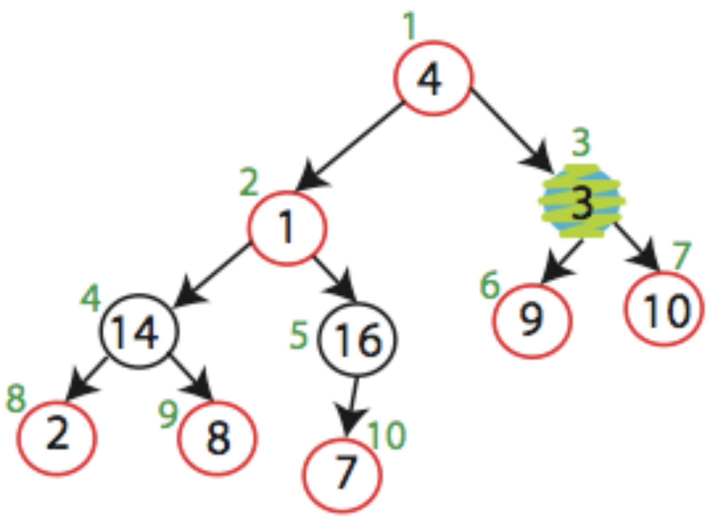
so heap property may only be violated at nodes 1…$\lfloor n/2 \rfloor$ of the tree

```
Build_Max_Heap(A):
    heap_size(A) = length(A)
    for i ← ⌊ length[A]/2⌋ downto 1
        do Max_Heapify(A, i)
```

Build_Max_Heap (Example Execution)

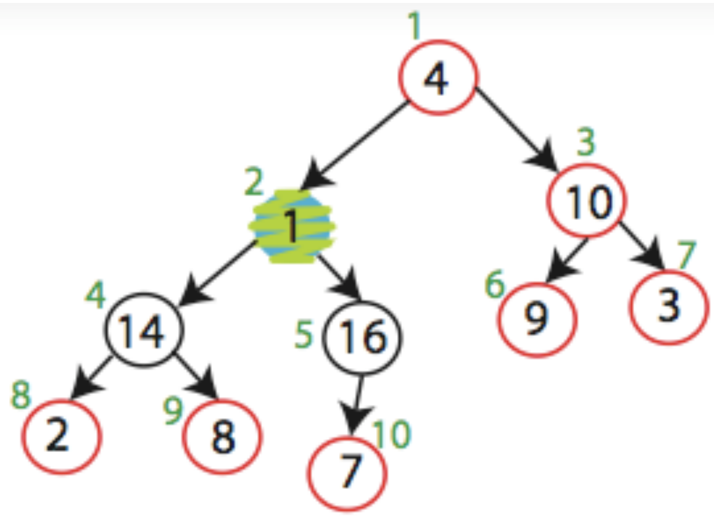


Build_Max_Heap (Example Execution)



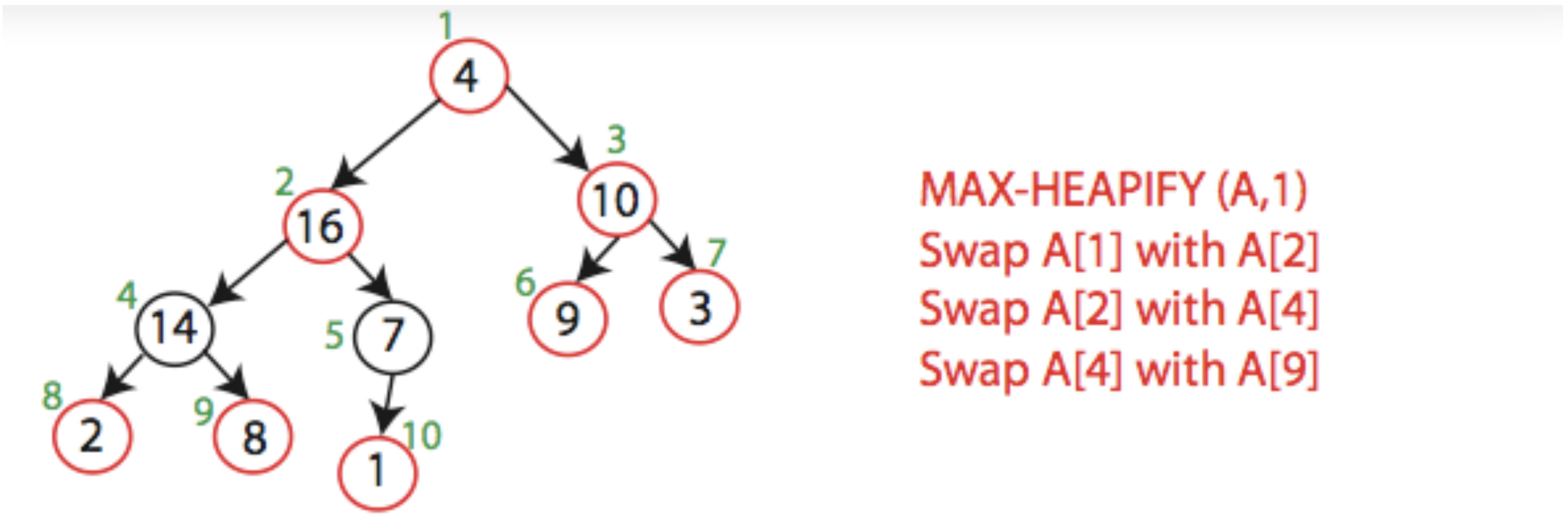
MAX-HEAPIFY (A,3)
Swap A[3] and A[7]

Build_Max_Heap (Example Execution)



MAX-HEAPIFY (A,2)
Swap A[2] and A[5]
Swap A[5] and A[10]

Build_Max_Heap (Example Execution)



Running Time: Trivially O(n log n), since I need to Heapify O(n) times.

Observe, however, that Heapify only pays O(1) time for the nodes that are one level above the leaves, and in general O(ℓ) for the nodes that are ℓ levels above the leaves. → O(n) time overall!

Heapsort

Recall Naïve Algorihm..

Sorting Strategy:

Find largest element of array, place it in last position; then find the largest among the remaining elements, and place it next to the largest, etc...

In notation:

$O(n^2)$

1. last_element = n;
2. Find maximum element A[i] of array A[1…last_element];
3. Swap A[i] and A[last_element];
4. last_element = last_element - 1;
5. Go to step 2

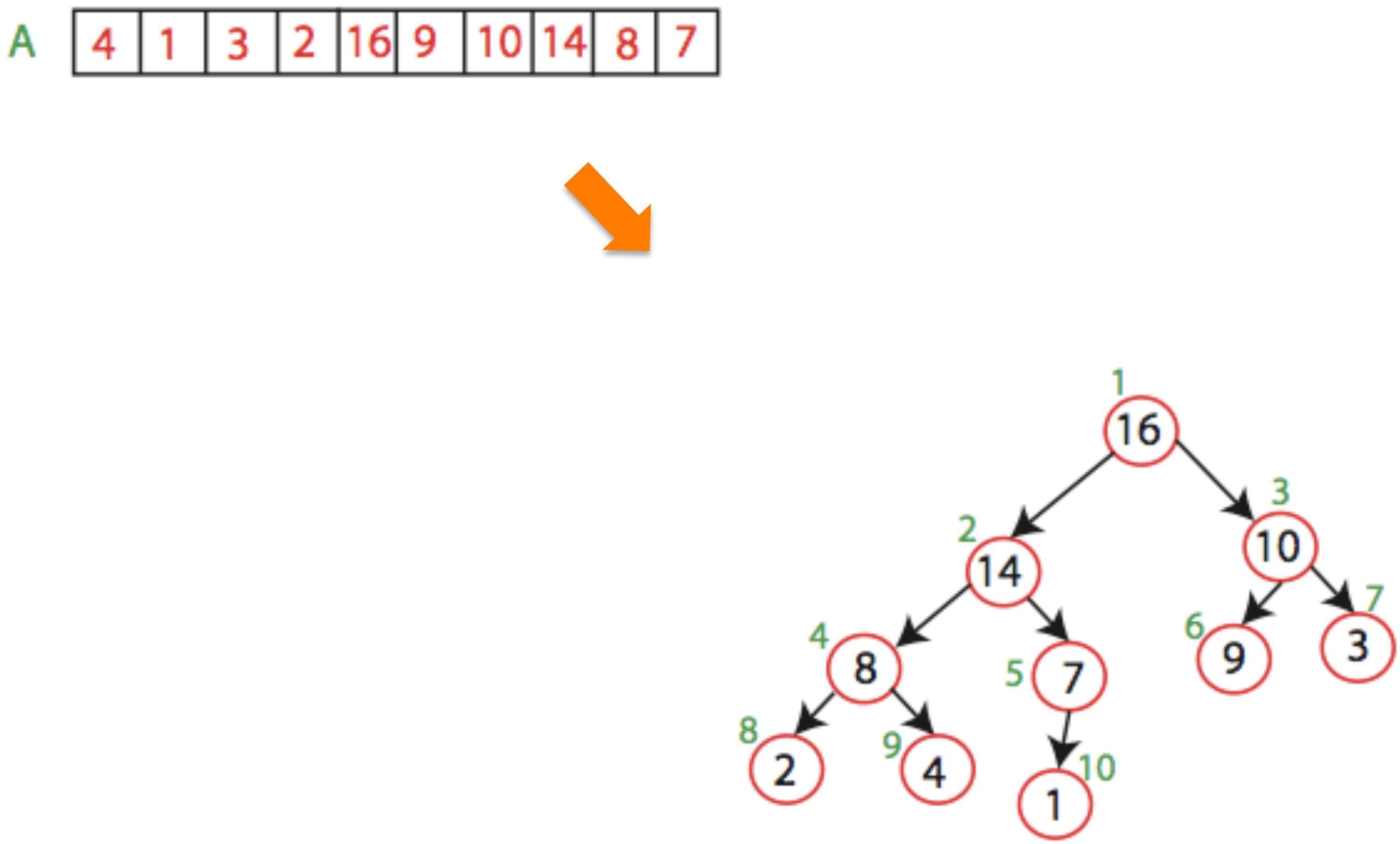
We have a fast data structure for step 2!
(which is also the most costly)

Heap-Sort

Sorting Strategy:

1. Build Max Heap from unordered array;

Heap-Sort



Heap-Sort

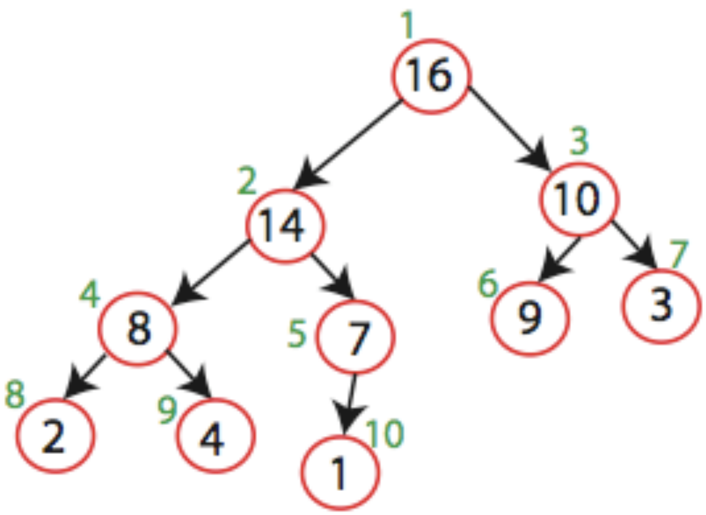
Sorting Strategy:

1. Build Max Heap from unordered array;

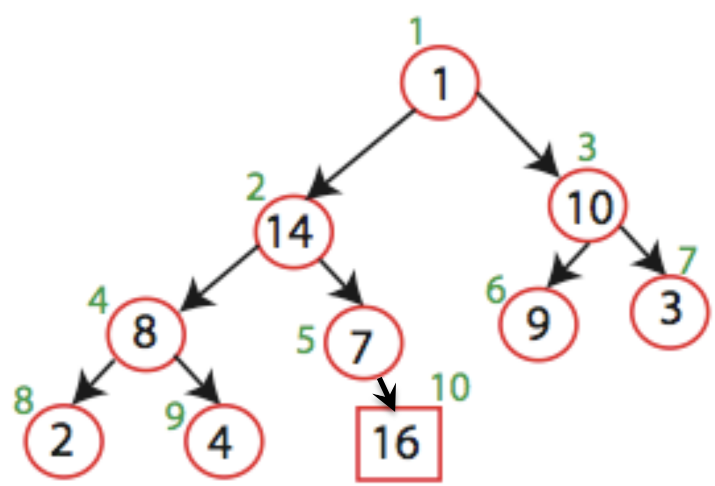
2. Find maximum element; this is A[1];

3. Swap elements A[n] and A[1]:
   now max element is at the end of the array!

Heap-Sort



Swap elements A[10] and A[1]



Heap-Sort

Sorting Strategy:

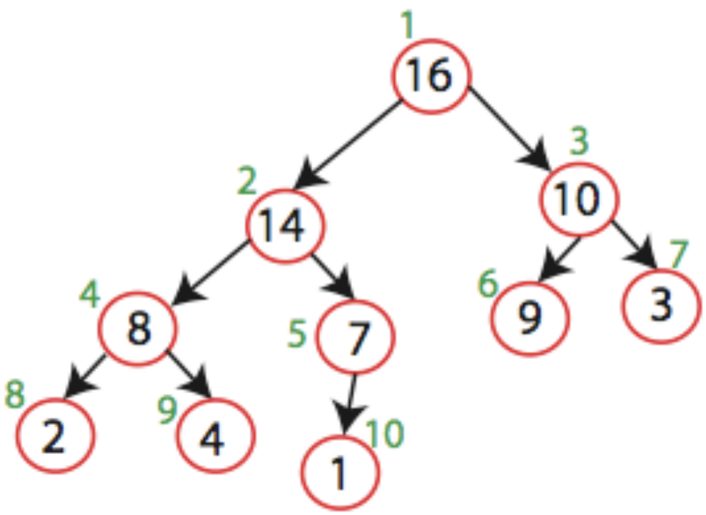
1. Build Max Heap from unordered array;

2. Find maximum element A[1];

3. Swap elements A[*n*] and A[1]:
   now max element is at the end of the array!

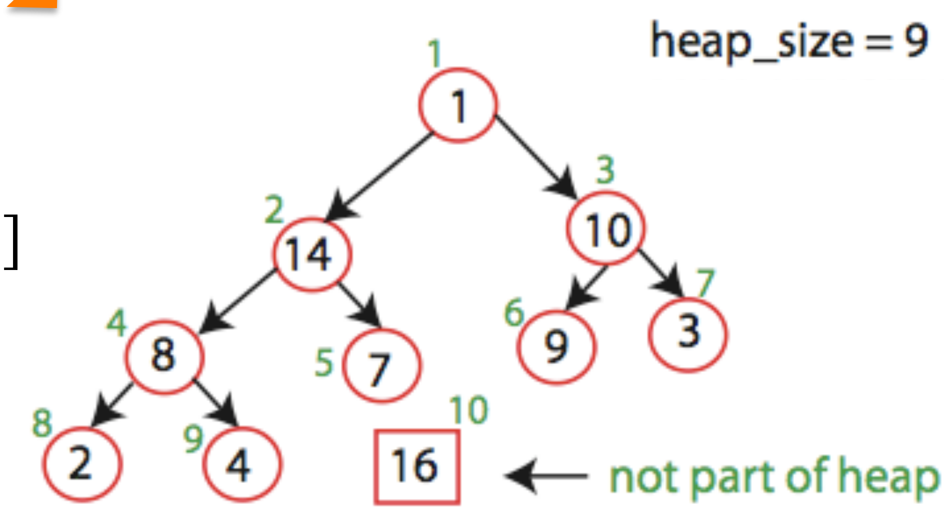
4. Discard node *n* from heap
   (by decrementing heap-size variable)

Heap-Sort



Swap elements A[10] and A[1]

heap_size = heap_size-1



Heap-Sort

Sorting Strategy:

1. Build Max Heap from unordered array;

2. Find maximum element A[1];

3. Swap elements A[n] and A[1]:
   now max element is at the end of the array!

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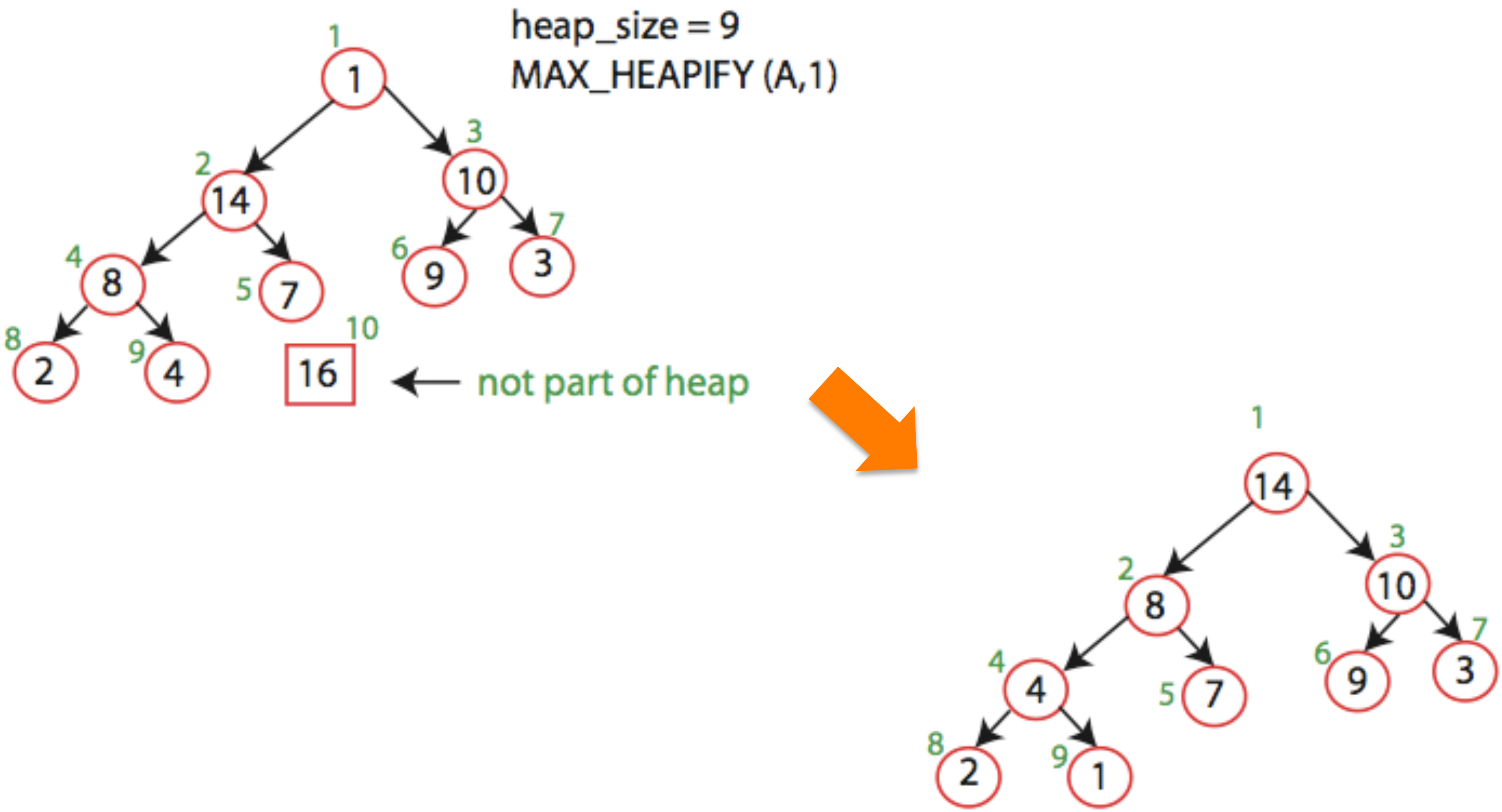
5. New root may violate max heap property, but its children are max heaps. Run max_heapify to fix this.

Heap-Sort



Heap-Sort

Sorting Strategy:

1. Build Max Heap from unordered array;

2. Find maximum element A[1];

3. Swap elements A[*n*] and A[1]:
   now max element is at the end of the array!

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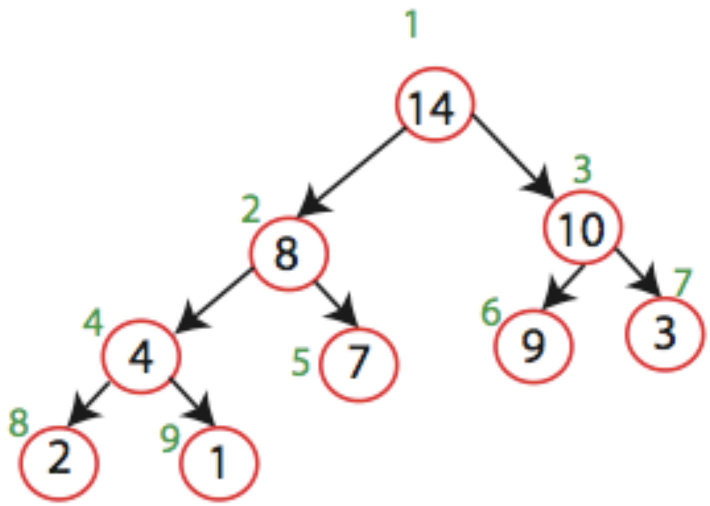
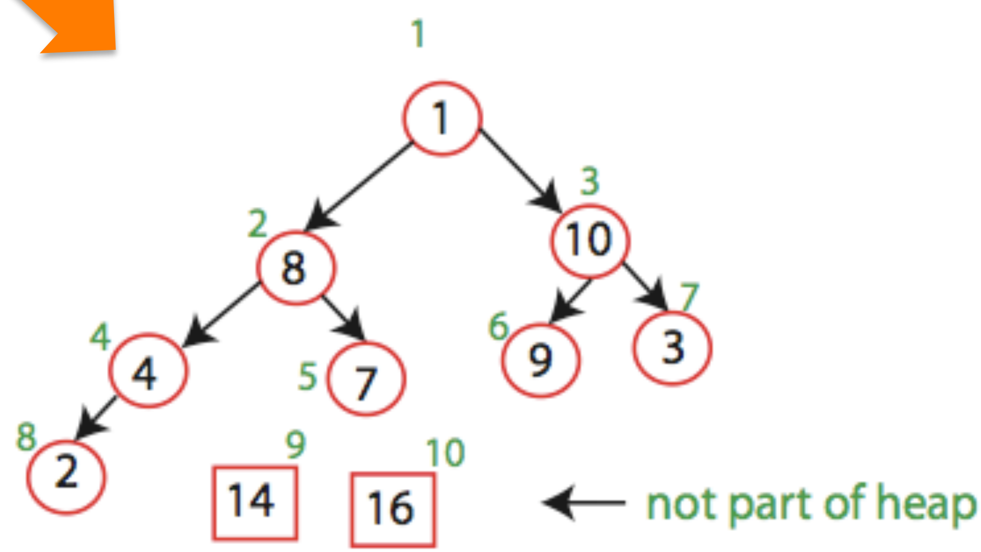
6. Go to step 2.

Heap-Sort

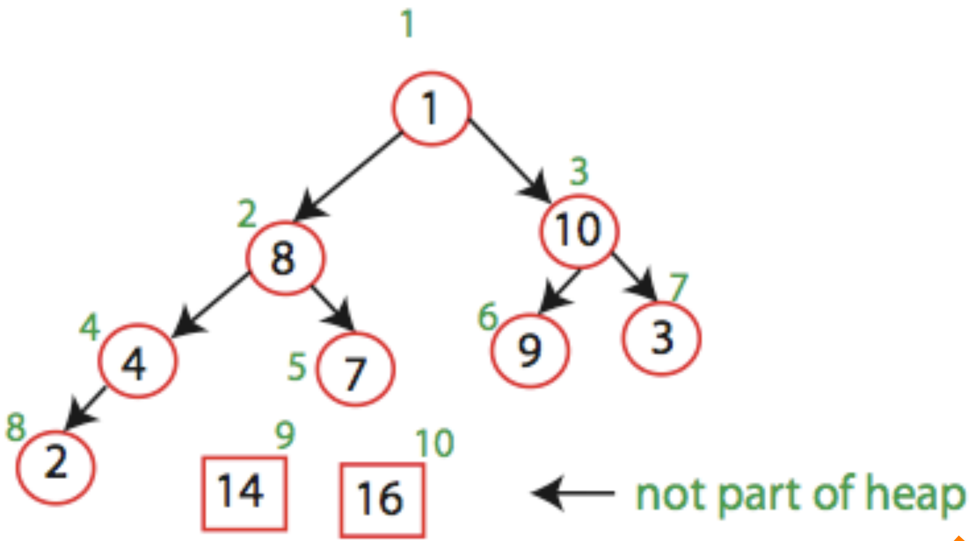


Swap elements A[9] and A[1]



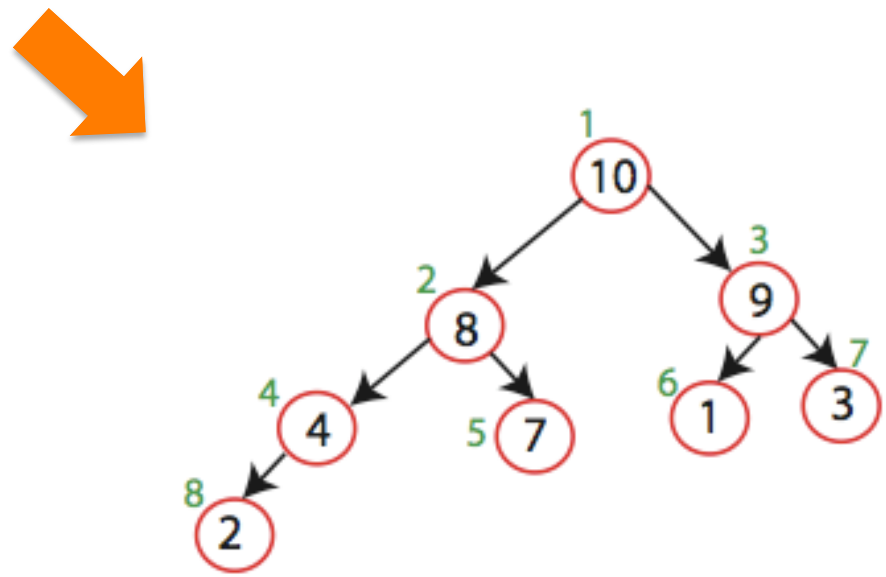
MAX_HEAPIFY (A,1)

Heap-Sort

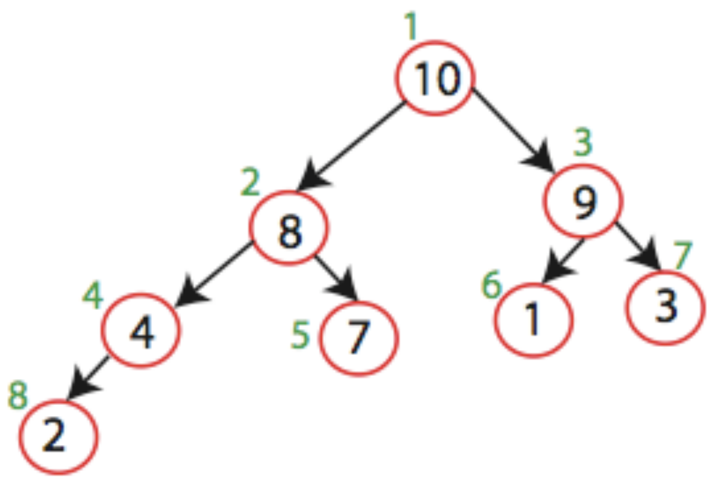


MAX_HEAPIFY (A,1)

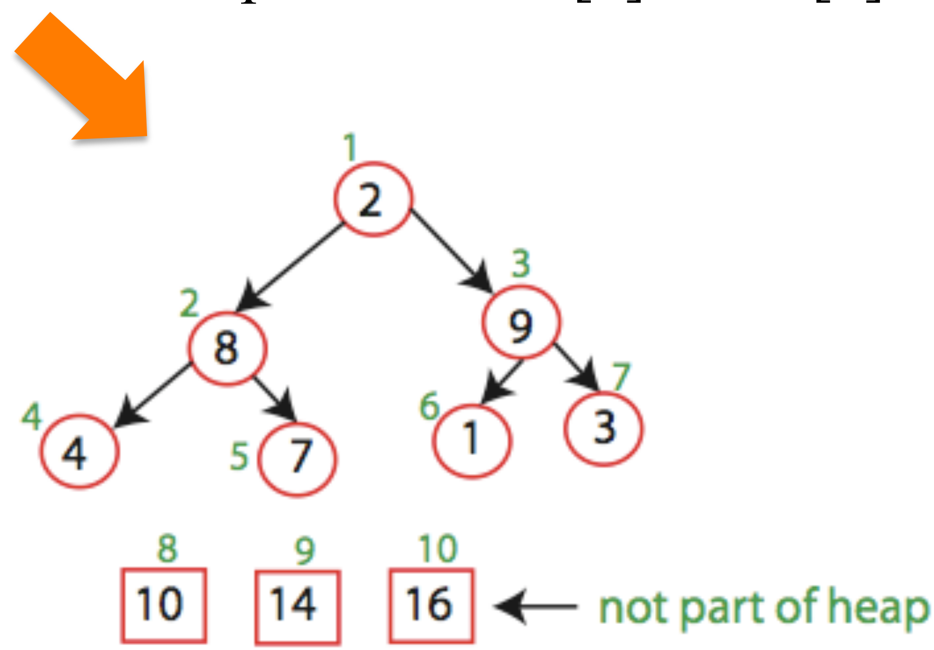
Max_Heapify(A,1)



Heap-Sort



Swap elements A[8] and A[1]



and so on…

Heap-Sort

Running time:

after n iterations the Heap is empty

every iteration involves a swap and a heapify operation; hence it takes $O(\log n)$ time

Overall $O(n \log n)$

Discussion: Other operations?

Operations with Heaps

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