Lecture 16: Shortest Paths III - Dijkstra and Special Cases

Lecture Overview

- Shortest paths in DAGs
- Shortest paths in graphs without negative edges
- Dijkstra's Algorithm

Readings

CLRS, Sections 24.2-24.3

DAGs:

Can't have negative cycles because there are no cycles!

1. Topologically sort the DAG. Path from u to v implies that u is before v in the linear ordering
2. One pass over vehicles in topologically sorted order relaxing each edge that leaves each vertex
   $\Theta(V + E)$ time

Example:

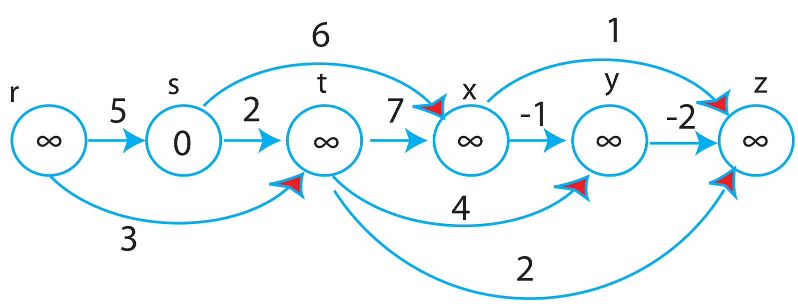
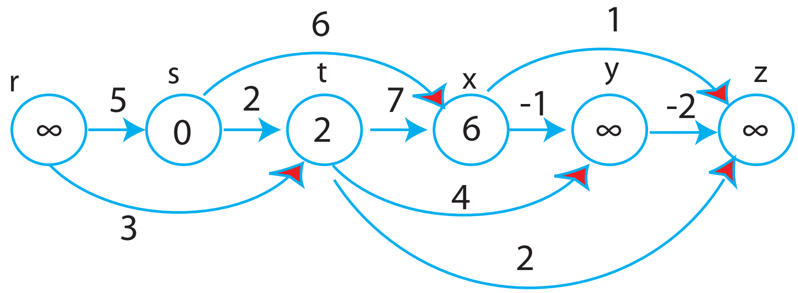


Figure 1: Shortest Path using Topological Sort.

Vertices sorted left to right in topological order

Process r: stays ∞. All vertices to the left of s will be ∞ by definition

Process s: $t : \infty \rightarrow 2$ $\quad$ $x : \infty \rightarrow 6$ (see top of Figure 2)



process t, x, y

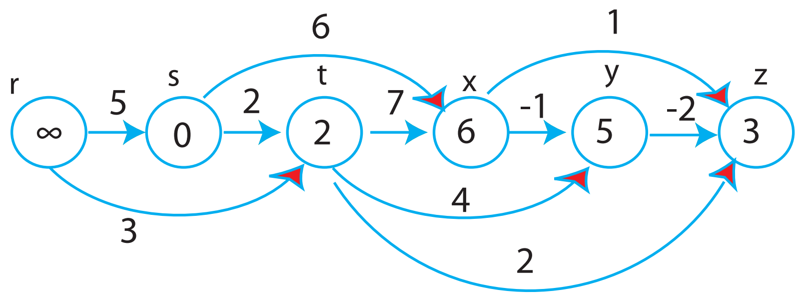


Figure 2: Preview of Dynamic Programming

DIJKSTRA Demo

Dijkstra's Algorithm

For each edge $(u, v) \epsilon E$, assume $w(u, v) \geq 0$, maintain a set S of vertices whose final shortest path weights have been determined. Repeatedly select $u \epsilon V - S$ with <u>minimum</u> shortest path estimate, add u to S, relax all edges out of u.

Pseudo-code

Dijkstra (G, W, s) //uses priority queue Q

  Initialize (G, s)

  $S \leftarrow \phi$

  $Q \leftarrow V[G]$ //Insert into Q

  while $Q \neq \phi$

    do $u \leftarrow$ EXTRACT-MIN(Q) //deletes u from Q

    $S = S \cup \{u\}$

    for each vertex $v \epsilon$ Adj$[u]$

      do RELAX (u, v, w) $\leftarrow$ this is an implicit DECREASE_KEY operation

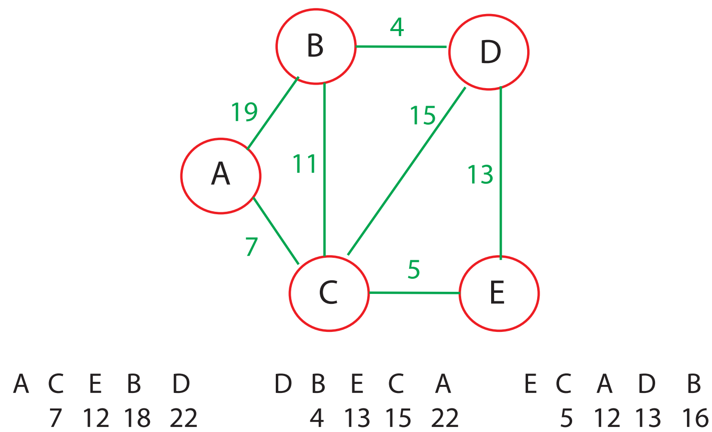


Figure 3: Dijkstra Demonstration with Balls and String.

Recall

RELAX(u, v, w)
if $d[v] > d[u] + w(u, v)$
then $d[v] \leftarrow d[u] + w(u, v)$
$\Pi[v] \leftarrow u$

Example

Strategy: Dijkstra is a greedy algorithm: choose closest vertex in $V - S$ to add to set S

Correctness: Each time a vertex u is added to set S, we have $d[u] = \delta(s, u)$

Complexity

$\theta(v)$ inserts into priority queue
$\theta(v)$ EXTRACT_MIN operations
$\theta(E)$ DECREASE_KEY operations

Array impl:

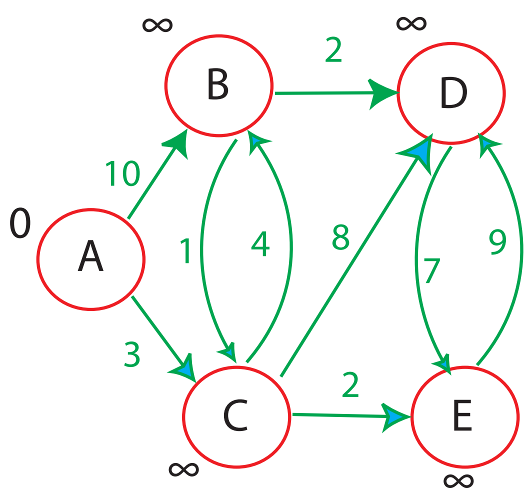
$\theta(v)$ time for extra min
$\theta(1)$ for decrease key
Total: $\theta(V.V + E.1) = \theta(V^2 + E) = \theta(V^2)$

Binary min-heap:

$\theta(\lg V)$ for extract min
$\theta(\lg V)$ for decrease key
Total: $\theta(V \lg V + E \lg V)$

Fibonacci heap (not covered in 6.006):

$\theta(\lg V)$ for extract min
$\theta(1)$ for decrease key
amortized cost
Total: $\theta(V \lg V + E)$



S = { }	{ A	B	C	D	E }	=	Q
S = { A }	0	∞	∞	∞	∞		
S = { A, C }	0	10	3	∞	∞	←	after relaxing edges from A
S = { A, C }	0	7	3	11	5	←	after relaxing edges from C
S = { A, C, E }	0	7	3	11	5		
S = { A, C , E, B}	0	7	3	9	5	←	after relaxing edges from B

Figure 4: Dijkstra Execution