In [23]:
```
## Includes
using ValidatedNumerics
using PyPlot
using NLopt
using ForwardDiff
using Interact
```

Deterministic Global Optimization for Continuous Functions

Harry Watson, 18.337 Final Project, Fall 2016

Despite the prevalence of optimization in the science and engineering community, comparatively little attention is given to deterministic methods for global optimization. The most commonly used "global" methods are multistart and meta-heuristic methods, which, while often effective, offer no guarantee of global optimality for nonconvex functions.

Instead, most algorithms for deterministic global optimization are based on the branch-and-bound framework (Horst and Tuy, 1996), which is implemented in this notebook.

The Branch-and-Bound Algorithm

The problems that can be solved by a standard implementation of this algorithm are of the form:

$$\min_{\mathbf{x}\in X} f(\mathbf{x}) \text{ s.t. } \quad \mathbf{g}(\mathbf{x}) \leq \mathbf{0},\ \mathbf{h}(\mathbf{x}) = \mathbf{0}$$

where $f : X \to \mathbb{R}$, $\mathbf{g} : X \to \mathbb{R}^{n_g}$ and $\mathbf{h} : X \to \mathbb{R}^{n_h}$. We'll assume these functions are continuously differentiable and that X is a compact set.

A global solution of the (possibly) nonconvex nonlinear program is found by bounding the optimal objective function value between lower and upper bounds, starting on an initial simple set such as a multidimensional interval (box) that defines the full search space of the problem. After generating initial bounds, this initial set is then partitioned into smaller sets according to some heuristic, which is known as branching. The bounding subroutines are then run on these smaller sets to yield tighter bounds. Any feasible point for the problem can serve as an upper bound, although in practice, it is generally preferable to use a local optimization algorithm to locate a local minimum of the objective function. Lower bounds are generated by constructing convex underestimators of the original program, which can be solved to global optimality with local optimization algorithms to yield a valid bound. As this procedure of branching and bounding continues, any set on which the lower bound is greater than the best known upper bound can be discarded, and no further branching is performed on this set (this is called fathoming by value dominance). Subsets of the search space may also be fathomed due to infeasibility of the lower bounding problem. The procedure continues until the best feasible solution value and the current lower bound agree to within a preset tolerance.

Branch and bound using intervals

The simplest "convex relaxation" we can generate on a given box is the constant bound given by evaluating the interval extension of our functions on the box and taking its the lower bound. Luckily, ValidatedNumerics.jl exists and provides the necessary interval arithmetic. We will use this strategy in the initial implementation of the branch-and-bound algorithm.

The next code block defines a type for a node in the branch-and-bound tree and an options structure for the user.

In [24]:
```
type babNode
    box::Vector{Interval{Float64}} ## Subproblem domain
    LB::Real ## Lower bound on subproblem
end

import Base.<
function <(node1::babNode, node2::babNode)
    if (node1.LB < node2.LB)
        return true;
    else
        return false;
    end
end

type options
    lbMethod::AbstractString ## Change lower bounding method ("Intervals" or
 "McCormick")
    rtol::Real        ## Relative termination tolerance
    atol::Real        ## Absolute termination tolerance
    maxNodes::Integer       ## Max nodes kept in memory at one time
    maxIterations::Integer  ## Max # of BaB iterations
    numIneqCons::Integer ## Number of inequality constraints
    numEqCons::Integer ## Number of equality constraints
end

options() = options("Intervals",1e-2,1e-8,10^5,10^5,0,0); ## Defaults
```

```
WARNING: Method definition (::Type{
```

Out[24]: options

```
Main.babNode})(Array{ValidatedNumerics.Interval{Float64}, 1}, Real) in module
 Main at In[2]:2 overwritten at In[24]:2.
WARNING: Method definition (::Type{Main.babNode})(Any, Any) in module Main at
 In[2]:2 overwritten at In[24]:2.
WARNING: Method definition <(Main.babNode, Main.babNode) in module Main at In
[2]:8 overwritten at In[24]:8.
WARNING: Method definition (::Type{Main.options})(AbstractString, Real, Real,
 Integer, Integer, Integer, Integer) in module Main at In[2]:16 overwritten a
t In[24]:16.
WARNING: Method definition (::Type{Main.options})(Any, Any, Any, Any, Any, An
y, Any) in module Main at In[2]:16 overwritten at In[24]:16.
WARNING: Method definition (::Type{Main.options})() in module Main at In[2]:2
5 overwritten at In[24]:25.
```

Below are the generic subroutines for the upper and lower bounding problems. The upper bounding problem is solved using NLopt.jl, and gradient information is generated automatically using ForwardDiff.jl.

In [25]:

```
## Interval version of lower-bounding problem
function LB_problem(probSize::Vector{Int64}, obj::Function, conIneq::Function,
 conEq::Function, box::Vector{Interval{Float64}})

    nx = probSize[1];
    nIneq = probSize[2];
    nEq = probSize[3];

    # Take solution as midpoint of box
    solution = Vector{Float64}(nx);
    for i = 1:nx
        solution[i] = (box[i].lo+box[i].hi)/2.0;
    end

    # Evaluate lower bound of objective on current box
    obj = obj(box);
    solutionValue = obj.lo;
    status = true;

    # Check constraint feasibility
    if nIneq > 0
        conIneq = conIneq(box);
        for i = 1:nIneq
            if conIneq[i].lo > 0.0
                solutionValue = -1e50;
                status = false;
            end
        end
    end
    if nEq > 0
        conEq = conEq(box);
        for i = 1:nEq
            if (conEq[i].lo > 0.0 || conEq[i].hi < 0.0)
                solutionValue = -1e50;
                status = false;
            end
        end
    end

    return solution, solutionValue, status;
end


## Upper bounding problem
function UB_problem(probSize::Vector{Int64}, obj::Function, conIneq::Function,
 conEq::Function, box::Vector{Interval{Float64}}, x0::Vector{Float64})

    nx = probSize[1];
    nIneq = probSize[2];
    nEq = probSize[3];

    opt = Opt(:LD_SLSQP, nx)

    lb = Vector{Float64}(nx);
    ub = Vector{Float64}(nx);
    for i=1:nx
        lb[i] = box[i].lo;
```

```
        ub[i] = box[i].hi;
    end

    lower_bounds!(opt,lb);
    upper_bounds!(opt, ub);

    xtol_rel!(opt,1e-4);

    min_objective!(opt,(x,g)->objWrapperUB(x,g,obj));

    if nIneq > 0
        tol = 1.0e-6*ones(nIneq);
        inequality_constraint!(opt, (f,x,g)->conIneqWrapperUB(f,x,g,conIneq),

    end
    if nEq > 0
        tol = 1.0e-6*ones(nEq);
        equality_constraint!(opt, (f,x,g)->conEqWrapperUB(f,x,g,conEq), tol);
    end

    (solutionValue,solution,ret) = optimize!(opt, x0);

    status = (ret==:XTOL_REACHED ? true : false);

    return solution, solutionValue, status;
end

## Wrappers for providing objective and constraint info to NLopt
function objWrapperUB(x::Vector, grad::Vector, obj::Function)
    g = x -> ForwardDiff.gradient(obj, x);
    if length(grad) > 0
        grad = g(x);
    end
    return obj(x);
end

function conIneqWrapperUB(result::Vector, x::Vector, grad::Matrix, conIneq::Fu
nction)
    g = x -> ForwardDiff.jacobian(conIneq, x);
    if length(grad) > 0
        grad = g(x)';
    end
    result = conIneq(x);
end

function conEqWrapperUB(result::Vector, x::Vector, grad::Matrix, conEq::Functi
on)
    g = x -> ForwardDiff.jacobian(conEq, x);
    if length(grad) > 0
        grad = g(x)';
    end
    result = conEq(x);
end
```

```
WARNING: Method definition LB_problem(Array{Int64, 1}
```

```
Out[25]: conEqWrapperUB (generic function with 1 method)
```

```
, Function, Function, Function, Array{ValidatedNumerics.Interval{Float64},
 1}) in module Main at In[3]:4 overwritten at In[25]:4.
WARNING: Method definition UB_problem(Array{Int64, 1}, Function, Function, Fu
nction, Array{ValidatedNumerics.Interval{Float64}, 1}, Array{Float64, 1}) in
 module Main at In[3]:46 overwritten at In[25]:46.
WARNING: Method definition objWrapperUB(Array{T<:Any, 1}, Array{T<:Any, 1}, F
unction) in module Main at In[3]:84 overwritten at In[25]:84.
WARNING: Method definition conIneqWrapperUB(Array{T<:Any, 1}, Array{T<:Any,
 1}, Array{T<:Any, 2}, Function) in module Main at In[3]:92 overwritten at In
[25]:92.
WARNING: Method definition conEqWrapperUB(Array{T<:Any, 1}, Array{T<:Any, 1},
 Array{T<:Any, 2}, Function) in module Main at In[3]:100 overwritten at In[2
5]:100.
```

Below is the body of the branch-and-bound algorithm that manages the execution of the bounding subproblems.

In [26]:

```
## Main branch and bound algorithm
function bab(obj::Function, conIneq::Function, conEq::Function, X0::Vector{Int
erval{Float64}}, opt::options)
## Problem dimension
nx = length(X0);
nIneq = opt.numIneqCons;
nEq = opt.numEqCons;
probSize = [nx, nIneq, nEq];

## Initialize
rtol = opt.rtol; # Relative termination tolerance
atol = opt.atol; # Absolute termination tolerance
maxNodes = opt.maxNodes; # Max nodes in memory at one time
maxIterations = opt.maxIterations; # Max iterations before giving up

numIterations = 0;
numNodesRemaining = 1;
currentLowerBound = -1e50;
lowerBound = -1e50;
currentUpperBound = 1e50;
upperBound = 1e50;
currentSolution = Vector{Float64}(nx);
solution = Vector{Float64}(nx);
headNode = babNode(X0,lowerBound);
nodeList = Vector{babNode}(0); push!(nodeList,headNode);
currentNode = nodeList[1];
converged = false;
lbFeasible = false;
ubFeasible = false;
index = 1;

while (~converged && numNodesRemaining > 0 && numNodesRemaining < maxNodes &&
numIterations < maxIterations)
    # Pick node for current iteration (the one with the least lower bound)
    currentNode,index = findmin(nodeList);
    lowerBound = nodeList[index].LB;
    fathom = false;

    # Solve lower bounding problem
    if opt.lbMethod == "Intervals"
        ## Intervals
        currentSolution, currentLowerBound, lbFeasible = LB_problem(probSize,
 conIneq, conEq, currentNode.box);
    elseif opt.lbMethod == "McCormick"
        ## McCormick relaxations
        currentSolution, currentLowerBound, lbFeasible = LB_problem2(probSize,
 obj, conIneq, conEq, currentNode.box);
    else
        println("\nInvalid lower bounding method specified.\n")
        return;
    end
    # Solve upper bounding problem
    if (lbFeasible)
        currentSolution, currentUpperBound, ubFeasible = UB_problem(probSize,
 conIneq, conEq, currentNode.box,currentSolution);
            if (ubFeasible)
                if (currentUpperBound < upperBound)
```

```
                    # Update the solution if necessary and flag for fathoming
 by value dominance
                    upperBound = currentUpperBound;
                    solution = currentSolution;
                    fathom = true;
            end
            branchNode(nx, currentNode, currentLowerBound, nodeList); # Branch
 on current node
            numNodesRemaining += 2;
        end
    else
        ## Flag for fathom by infeasibility
        fathom = true;
    end

    # Delete visited node from list
    deleteat!(nodeList,index);
    numNodesRemaining -= 1;

    # Fathom tree
    if (fathom)
        numNodesFathomed = 0;
        numNodesFathomed = fathomTree(nodeList, upperBound);
        numNodesRemaining = numNodesRemaining - numNodesFathomed;
    end

    numIterations += 1;


        Abs_gap = upperBound-lowerBound;
        if (Abs_gap < rtol*abs(upperBound) || Abs_gap < atol)
            converged = true;
        end

        ## Print out when solution updates
        if (fathom && lbFeasible)
            println("\nIteration: ", numIterations,
                    "\nNodes in memory: ", numNodesRemaining,
                    "\nCurrent solution found at: ", solution, " with value =
 ", upperBound,
                    "\nBest possible solution value (lower bound): ", lowerBou
nd,
                    "\nThe bounds are ", max(100*max(0.0,1.0-(Abs_gap)),100*ma
x(0.0,1.0-(Abs_gap)/abs(upperBound))), "% converged.")

        end
        #@printf("%d || %d || %f || %f || %f\n", numIterations, numNodesRemain
ing, upperBound, lowerBound, 100*max(0.0,1.0-(Abs_gap)/abs(upperBound)) );
end

    if (numIterations >= maxIterations)
        println("Exceeded max number of iterations!")
    end
    if (numNodesRemaining >= maxNodes)
        println("Exceeded max number of nodes in memory!")
    end
    Abs_gap = upperBound-lowerBound;
```

```
    println("\nBest solution found at: ", solution, " with value = ", upperBou
nd,
        "\nBest possible solution value (lower bound): ", lowerBound,
        "\nThe bounds are ", max(100*max(0.0,1.0-(Abs_gap)),100*max(0.0,1.0-(A
bs_gap)/abs(upperBound))), "% converged.",
        "\nStatistics: ", numIterations," iterations");

end

function branchNode(nx::Integer, node::babNode, currentLB::Real, nodeList::Vec
tor{babNode})

    # Find largest diameter of current node's box (absolute sense)
    max_width = node.box[1].hi - node.box[1].lo;
    branch_index = 1;

    for i = 1:nx
        cur_width = (node.box[i].hi - node.box[i].lo);
        if (cur_width > max_width)
            max_width = cur_width;
            branch_index = i;
        end
    end

    #Bisect along that dimension
    tmp = node.box[branch_index];
    tmp1 = @interval(tmp.lo,(tmp.lo+tmp.hi)/2.0);
    tmp2 = @interval((tmp.lo+tmp.hi)/2.0, tmp.hi);

    box1 = Vector{Interval}(nx);
    box2 = Vector{Interval}(nx);
    for i=1:nx
        box1[i] = node.box[i];
        box2[i] = node.box[i];
    end

    box1[branch_index] = tmp1;
    box2[branch_index] = tmp2;


    node1 = babNode(box1,currentLB);
    node2 = babNode(box2,currentLB);

    push!(nodeList,node1);
    push!(nodeList,node2);

end

function fathomTree(nodeList::Vector{babNode},upperBound::Real)
    i = 1;
    numNodesFathomed = 0;
    while(i <= length(nodeList))
        if (nodeList[i].LB > upperBound)
            deleteat!(nodeList,i);
            numNodesFathomed += 1;
        end
        i += 1;
```

```
    end
    return numNodesFathomed;
end
```

```
WARNING: Method definition bab(Function,
```

Out[26]:
```
fathomTree (generic function with 1 method)
```

```
Function, Function, Array{ValidatedNumerics.Interval{Float64}, 1}, Main.optio
ns) in module Main at In[4]:4 overwritten at In[26]:4.
WARNING: Method definition branchNode(Integer, Main.babNode, Real, Array{Mai
n.babNode, 1}) in module Main at In[4]:134 overwritten at In[26]:114.
WARNING: Method definition fathomTree(Array{Main.babNode, 1}, Real) in module
 Main at In[4]:170 overwritten at In[26]:150.
```

Example: Six-Hump Camel Function

Find a global minimizer of:

$$f(\mathbf{x}) = (4 - 2.1x_1^2 + \frac{1}{3}x_2^4)x_1^2 + x_1x_2 + (-4 + 4x_2^2)x_2^2$$

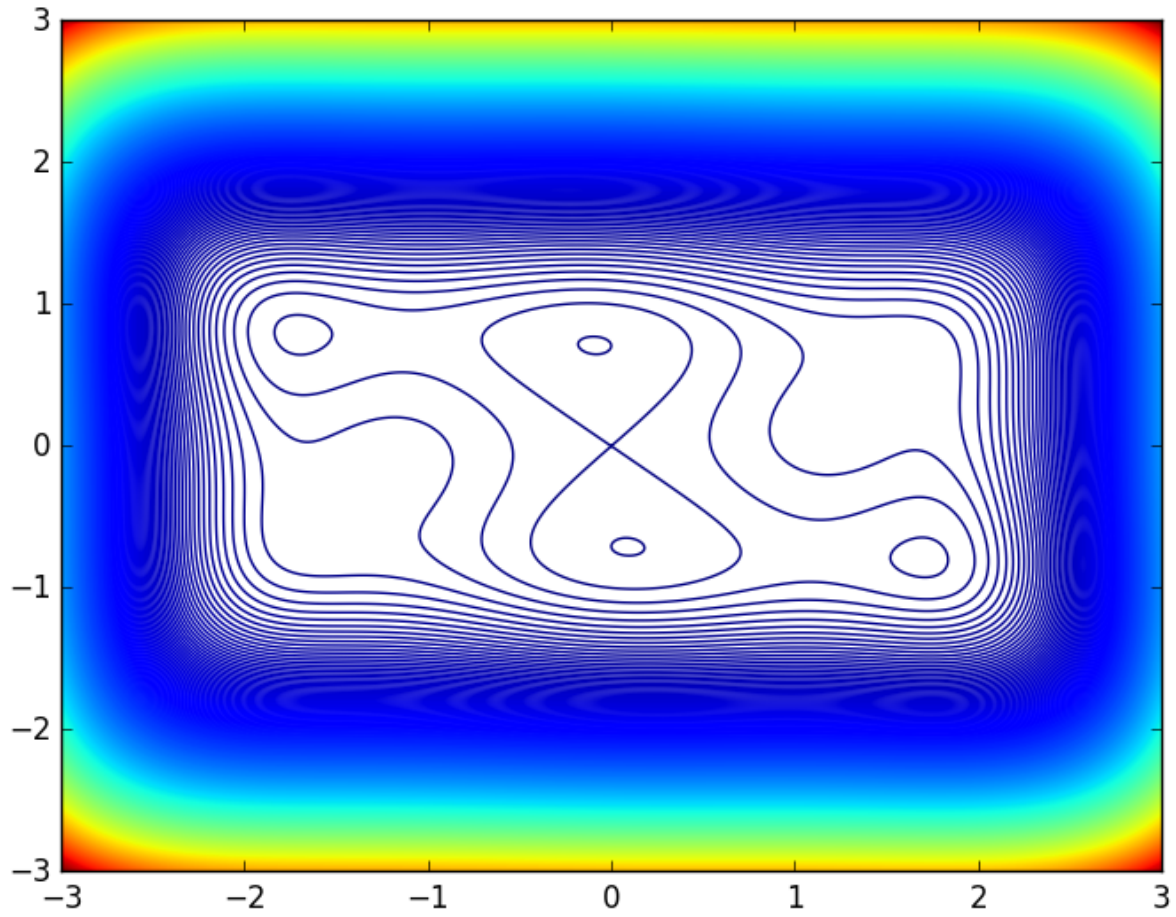
on the 2-dimensional box $X = [-3, 3]^2$ with no constraints.

In [27]:
```
function sixHump(x::Vector)
    return (4.0 - 2.1*x[1]^2+(1./3.)*x[1]^4)*x[1]^2 + x[1]*x[2]+(-4.0 + 4.0*x[
^2)*x[2]^2;
end

function noConstraint(x::Vector);
    return;
end


box = [(-3.0..3.0), (-3.0..3.0)]

## Plot the contours
xx = collect(-3.0:0.01:3.0);
zz = Matrix(length(xx),length(xx))
for i=1:length(xx)
    for j=1:length(xx)
        zz[i,j] = sixHump([xx[i],xx[j]])
    end
end
contour(xx,xx,zz',500)
```




```
WARNING: Method definition sixHump(Array
```

Out[27]: PyObject <matplotlib.contour.QuadContourSet object at 0x0000000036EC6EF0>

In [28]:
```
opt = options("Intervals",1e-2,1e-8,10^5,10^5,0,0);
@time bab(sixHump, noConstraint, noConstraint, box, opt)
```

```
{T<:

Iteration: 1
Nodes in memory: 2
Current solution found at: [0.0,0.0] with value = 0.0
Best possible solution value (lower bound): -1.0e50
The bounds are 0.0% converged.

Iteration: 209
Nodes in memory: 168
Current solution found at: [-0.375,0.75] with value = -0.7437263488769531
Best possible solution value (lower bound): -12.881250000000001
The bounds are 0.0% converged.

Any, 1}) in module Main at In[5]:2 overwritten at In[27]:2.
WARNING: Method definition noConstraint(Array{T<:Any, 1}) in module Main at I
n[5]:5 overwritten at In[27]:5.
```

```
Iteration: 785
Nodes in memory: 573
Current solution found at: [-0.1875,0.5625] with value = -0.8325981259346008
Best possible solution value (lower bound): -2.53125
The bounds are 0.0% converged.

Iteration: 812
Nodes in memory: 499
Current solution found at: [-0.09375,0.84375] with value = -0.864477026648819
4
Best possible solution value (lower bound): -2.42578125
The bounds are 0.0% converged.

Iteration: 885
Nodes in memory: 503
Current solution found at: [-0.09375,0.5625] with value = -0.882913458533585
Best possible solution value (lower bound): -2.21484375
The bounds are 0.0% converged.

Iteration: 941
Nodes in memory: 489
Current solution found at: [-0.09375,0.65625] with value = -1.007299292273819
5
Best possible solution value (lower bound): -2.07421875
The bounds are 0.0% converged.

Iteration: 1149
Nodes in memory: 572
Current solution found at: [-0.140625,0.703125] with value = -1.0204679281247
082
Best possible solution value (lower bound): -1.6442033411934973
The bounds are 38.877509436772385% converged.

Iteration: 1185
Nodes in memory: 535
Current solution found at: [-0.046875,0.703125] with value = -1.0240539336606
163
Best possible solution value (lower bound): -1.6083984375
The bounds are 42.93811247318126% converged.

Iteration: 1425
Nodes in memory: 600
Current solution found at: [-0.117188,0.703125] with value = -1.0277348784995
866
Best possible solution value (lower bound): -1.3871232630684973
The bounds are 65.03102190191404% converged.

Iteration: 1496
Nodes in memory: 539
Current solution found at: [-0.0703125,0.703125] with value = -1.029588249652
8521
Best possible solution value (lower bound): -1.351318359375
The bounds are 68.75157522139301% converged.

Iteration: 1820
Nodes in memory: 650
Current solution found at: [-0.0820313,0.726563] with value = -1.029667222376
```

```
6685
Best possible solution value (lower bound): -1.19919826630794
The bounds are 83.53535586575617% converged.

Iteration: 1848
Nodes in memory: 550
Current solution found at: [-0.105469,0.714844] with value = -1.0306742944702
258
Best possible solution value (lower bound): -1.190621894784272
The bounds are 84.48126617960718% converged.

Iteration: 1872
Nodes in memory: 504
Current solution found at: [-0.0820313,0.714844] with value = -1.031333871621
3942
Best possible solution value (lower bound): -1.1882625951908723
The bounds are 84.78390675535894% converged.

Iteration: 2252
Nodes in memory: 744
Current solution found at: [-0.0878906,0.714844] with value = -1.031570033893
7457
Best possible solution value (lower bound): -1.1160741582058564
The bounds are 91.80820288147156% converged.

Iteration: 3117
Nodes in memory: 1355
Current solution found at: [-0.0908203,0.711914] with value = -1.031619488608
5757
Best possible solution value (lower bound): -1.0697407247675583
The bounds are 96.30471927101729% converged.

Iteration: 4277
Nodes in memory: 2180
Current solution found at: [-0.0893555,0.711914] with value = -1.031623383582
6774
Best possible solution value (lower bound): -1.0528284560600296
The bounds are 97.94449478222273% converged.

Iteration: 6748
Nodes in memory: 4165
Current solution found at: [-0.0900879,0.713379] with value = -1.031624116353
1195
Best possible solution value (lower bound): -1.0422166495895508
The bounds are 98.97321775746413% converged.

Best solution found at: [-0.0900879,0.713379] with value = -1.031624116353119
5
Best possible solution value (lower bound): -1.0419374498355465
The bounds are 99.00028185470445% converged.
Statistics: 6888 iterations
  1.367986 seconds (3.81 M allocations: 121.090 MB, 11.06% gc time)
```

One issue with using natural interval extensions of functions in branch-and-bound is that the intervals only converge linearly to the true function as the width of the domain shrinks. To be precise, the definition of relaxation convergence order is given below.

Definition: Let $C \subset \mathbb{R}^n$ be a convex set and $\mathbb{I}C$ be the set of all interval subsets of C. Let a continuous convex relaxation of $f : C \to \mathbb{R}$ on any $X \in \mathbb{I}C$ be given by $f_X^{cv} : X \to \mathbb{R}$. The relaxations are said to have *convergence order* $\beta \geq 1$ if there exists $K > 0$ such that:

$$\min_{\mathbf{x} \in X} f(\mathbf{x}) - \min_{\mathbf{x} \in X} f_X^{cv}(\mathbf{x}) \leq K w(X)^{\beta}, \quad \forall X \in \mathbb{I}C,$$

where $w(X)$ is the width of an n-dimensional interval X, defined as $w(X) = \max_{i=1,\ldots,n}(X_i^U - X_i^L)$.

In unconstrained optimization, linear convergence leads to the branch-and-bound algorithm having to search a very large number of boxes in the vicinity of minima (the "cluster problem").

We see this effect manifest if we make the termination tolerance tighter:

In [29]:
```
opt = options("Intervals",5e-3,1e-8,10^5,10^5,0,0);
@time bab(sixHump, noConstraint, noConstraint, box, opt)
```

```
Iteration: 1
Nodes in memory: 2
Current solution found at: [0.0,0.0] with value = 0.0
Best possible solution value (lower bound): -1.0e50
The bounds are 0.0% converged.

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Best possible solution value (lower bound): -1.3871232630684973
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The bounds are 65.03102190191404% converged.

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Best possible solution value (lower bound): -1.351318359375
The bounds are 68.75157522139301% converged.

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Best possible solution value (lower bound): -1.19919826630794
The bounds are 83.53535586575617% converged.

Iteration: 1848
Nodes in memory: 550
Current solution found at: [-0.105469,0.714844] with value = -1.0306742944702
258
Best possible solution value (lower bound): -1.190621894784272
The bounds are 84.48126617960718% converged.

Iteration: 1872
Nodes in memory: 504
Current solution found at: [-0.0820313,0.714844] with value = -1.031333871621
3942
Best possible solution value (lower bound): -1.1882625951908723
The bounds are 84.78390675535894% converged.

Iteration: 2252
Nodes in memory: 744
Current solution found at: [-0.0878906,0.714844] with value = -1.031570033893
7457
Best possible solution value (lower bound): -1.1160741582058564
The bounds are 91.80820288147156% converged.

Iteration: 3117
Nodes in memory: 1355
Current solution found at: [-0.0908203,0.711914] with value = -1.031619488608
5757
Best possible solution value (lower bound): -1.0697407247675583
The bounds are 96.30471927101729% converged.

Iteration: 4277
Nodes in memory: 2180
Current solution found at: [-0.0893555,0.711914] with value = -1.031623383582
6774
Best possible solution value (lower bound): -1.0528284560600296
The bounds are 97.94449478222273% converged.

Iteration: 6748
Nodes in memory: 4165
Current solution found at: [-0.0900879,0.713379] with value = -1.031624116353
1195
Best possible solution value (lower bound): -1.0422166495895508
The bounds are 98.97321775746413% converged.
```

```
Iteration: 7272
Nodes in memory: 4367
Current solution found at: [-0.0900879,0.712646] with value = -1.031628214562
3794
Best possible solution value (lower bound): -1.041182579553759
The bounds are 99.07385578869297% converged.

Iteration: 11721
Nodes in memory: 8025
Current solution found at: [-0.0897217,0.712646] with value = -1.031628397423
9073
Best possible solution value (lower bound): -1.0369223827002936
The bounds are 99.48683214909497% converged.

Best solution found at: [-0.0897217,0.712646] with value = -1.031628397423907
3
Best possible solution value (lower bound): -1.0367837535010869
The bounds are 99.50027004975308% converged.
Statistics: 12032 iterations
  2.687499 seconds (6.42 M allocations: 201.838 MB, 3.38% gc time)
```

McCormick's Relaxations (McCormick, 1976)

I now describe and implement an alternative method for bounding the range of a function.

First introduced (though largely ignored for many years) by G.P. McCormick in 1976, this is an automatic method for constructing and evaluating convex underestimators and concave overestimators for finite compositions of:

- Addition
- Multiplication
- Univariate intrinsic functions (sin/exp/log/pow/etc.)

"McCormick arithmetic" can be thought of as an extension of interval arithmetic. Indeed, the McCormick convex/concave relaxations of a function are defined with respect to a specific interval domain, so an interval arithmetic library is needed (again, here I use ValidatedNumerics.jl).

These relaxations have the following attributes/properties:

- Computationally inexpensive (scalar multiple of function evaluation)
- Subgradients are computable using automatic differentiation techniques
- Converge quadratically to original function as domain shrinks (no clustering)
- Not necessarily differentiable

For this implementation, we think of a *McCormick object* $\mathcal{Z}$ as an extension of an interval object, i.e.:

$$\mathcal{Z} = (Z^B, Z^C) = ([z^L, z^U], [z^{cv}, z^{cc}]).$$

In implementation, it is most convenient to represent a McCormick object as an interval and a pair of real numbers (and associated derivative information), which leads to the following type:

In [8]:
```
type mc{T<:Real, U<:Integer}
    I::Interval{T}   # Bounding interval
    cv::T            # Convex relaxation
    cc::T            # Concave relaxation
    cvsub::Vector{T} # Gradient of convex relaxation
    ccsub::Vector{T} # Gradient of concave relaxation
    nsub::U          # Length of gradient vector
    constant::Bool   # Are relaxations constant (i.e. just an interval)?

    mc(interval,cv,cc,cvsub,ccsub,nsub,constant) = (length(cvsub) == nsub && l
ength(ccsub) == nsub)? new(interval,cv,cc,cvsub,ccsub,nsub,constant) :
error("Inconsistent subgradient size!");
end
```

In [9]:
```
## Constructors
mc() = mc(@interval(0.0),0.0,0.0,Vector{Float64}(0),Vector{Float64}
(0),0,true);
mc{U<:Integer}(nsub::U) =
mc(@interval(0.0),0.0,0.0,zeros(Float64,nsub),zeros(Float64,nsub),nsub,nsub>0
? false : true);
mc{T<:Real}(I::Interval{T}) = mc(I,I.lo,I.hi,Vector{Float64}
(0),Vector{Float64}(0),0,true);
mc{T<:Real}(I::Interval{T},c::T) = mc(I,c,c,Vector{Float64}(0),Vector{Float64}
(0),0,false);
mc{T<:Real}(I::Interval{T},cv::T,cc::T) = mc(I,cv,cc,Vector{Float64}(0),Vector
oat64}(0),0,false);
mc{T<:Real,U<:Integer}(I::Interval{T},c::T,nsub::U) = mc(I,c,c,zeros(T,nsub),z
eros(T,nsub),nsub,false);
mc{T<:Real}(I::Interval{T},cv::T,cc::T,cvsub::Vector{T},ccsub::Vector{T}) = mc
,cv,cc,cvsub,ccsub,length(cvsub),false);
mc{T<:Real,U<:Integer}(I::Interval{T},cv::T,cc::T,nsub::U) = mc(I,cv,cc,zeros(
nsub),zeros(T,nsub),nsub,false);
mc{T<:Real,U<:Integer}(I::Interval{T},cv::T,cc::T,cvsub::Vector{T},ccsub::Vect
or{T},nsub::U) = mc{T,U}(I,cv,cc,cvsub,ccsub,nsub,false);
mc{T<:Real,U<:Integer}(I::Interval{T},cv::T,cc::T,cvsub::Vector{T},ccsub::Vect
or{T},nsub::U,constant::Bool) = mc{T,U}(I,cv,cc,cvsub,ccsub,nsub,constant);
```

The next block of code implements the McCormick relaxations and their subgradients using operator overloading. It relies on Validated Numerics.jl as the underlying interval library.

The basic rules for addition and multiplication are as follows:

$$+(\mathcal{X},\mathcal{Y}) = (X^B + Y^B, X^C + Y^C),$$
$$\times(\mathcal{X},\mathcal{Y}) = (X^B Y^B, [z^{cv}, z^{cc}]),$$

where: \begin{align} z^{cv}&=\max\left(\left(y^LX^C+x^LY^C-x^Ly^L\right)^L, \left(y^UX^C+x^UY^C-x^U\,y^U\right)^L, (X^BY^B)^L\right),\ z^{cc}&=\min\left(\left(y^LX^C+x^UY^C-x^Uy^L\right)^U, \left(y^UX^C+x^LY^C-x^Ly^U\right)^U, (X^BY^B)^U\right). \end{align}

(Note that the product function is implemented equivalently but without using max/min below for ease of setting correct subgradients, in the style of the C++ code developed in my research group.)

Various univariate functions such as exp, log, sqrt, etc. also have "rules" for the construction of their convex/concave relaxation. For most of these, the function itself is either the convex or concave relaxation, and the other relaxation is given by the line segment that joins the function value at each end of the interval.

Finally, there is a rule for finite compositions of such functions:

Taking $u^{cv}(X,\cdot)$ and $u^{cc}(X,\cdot)$ which are known convex and concave relaxations of the univariate function u on X, and $x^{\min}(X)$ and $x^{\max}(X)$ which are a minimum of $u^{cv}(X,\cdot)$ and a maximum of $u^{cc}(X,\cdot)$ on X, then (assuming all the domains/ranges are compatible), the composite convex relaxation $u(\mathcal{X})$ is given by:
$u(\mathcal{X}) = \left(u(X^B), \left[u^{cv}(X^B, \mathrm{mid}(x^{cv}, x^{cc}, x^{\min}(X^B))), u^{cc}(X^B, \mathrm{mid}(x^{cv}, x^{cc}, x^{\max}(X^B)))\right] \cap u(X^B)\right)$.

This result is implemented in the various univariate functions below.

In [10]:

```
## Helper functions

function cut(MC1::mc) ## cut relaxations at interval bounds
    if MC1.cv < MC1.I.lo
        MC1.cv = MC1.I.lo;
        for i = 1:MC1.nsub
            MC1.cvsub[i] = 0.0;
        end
    end
    if MC1.cc > MC1.I.hi
        MC1.cc = MC1.I.hi;
        for i = 1:MC1.nsub
            MC1.ccsub[i] = 0.0;
        end
    end
    return MC1;
end

function mid(a::Real, b::Real, c::Real)
  ## Return the mid value of three scalars; e.g., b if a <= b <= c
    if(( b <= a && a <= c ) || ( c <= a && a <= b ))
        return a;
    end
    if(( a <= b && b <= c ) || ( c <= b && b <= a ))
        return b;
    end
        return c;
 end

function mid(cv::Real, cc::Real, cut::Real, id::Integer)
    ## Return the mid value of three scalars cv, cc and cut, knowing that CONV
 <= CONC
    ## If the argument id is negative, the function indicates which value is t
he mid
    if ( id < 0 )
        if ( cut < cv )
            id = 1;
            return cv;
        elseif ( cut > cc )
            id = 2;
            return cc;
        else
            id = 0;
            return cut;
        end
    end

    ## If the argument id is nonnegative,the function returns the correspondin
g value, which is not necessarily the mid value
    if ( id == 1 )
        return cv;
    elseif ( id == 0  )
        return cut;
    else
        return cc;
    end
end
```

```
function mid( dcv::Vector, dcc::Vector, k::Integer, id::Integer )
    if( id == 1 )
        return dcv[k];
    elseif ( id == 0  )
        return 0.;
    else
        return dcc[k];
    end
end

function mid( dcv::Vector, dcc::Vector, dcut::Vector, k::Integer, id::Integer
)
    if( id == 1 )
        return dcv[k];
    elseif ( id == 0  )
        return dcut[k];
    else
        return dcc[k];
    end
end

## Operator overloads

## Negation
import Base.-
function -(MC1::mc)
    mcOut = mc(MC1.nsub);

    mcOut.I = (-MC1.I);
    mcOut.cv = -MC1.cc;
    mcOut.cc = -MC1.cv;

    if(!mcOut.constant)
        mcOut.cvsub = -MC1.ccsub;
        mcOut.ccsub = -MC1.cvsub;
    end
    return mcOut;
end

## Addition
import Base.+
function +(MC1::mc,x::Real)
    mcOut = mc(MC1.nsub);
    mcOut.I = MC1.I + x;
    mcOut.cv = MC1.cv + x;
    mcOut.cc = MC1.cc + x;
    mcOut.cvsub = MC1.cvsub;
    mcOut.ccsub = MC1.ccsub;
    return mcOut;
end

import Base.+
function +(x::Real,MC1::mc)
    return MC1+x;
end
```

```
function sum1(MC1::mc, MC2::mc) ## MC2 constant
    mcOut = mc(MC1.nsub);
    mcOut.I = MC1.I + MC2.I;
    mcOut.cv = MC1.cv + MC2.cv;
    mcOut.cc = MC1.cc + MC2.cc;
    mcOut.cvsub = MC1.cvsub;
    mcOut.ccsub = MC1.ccsub;
    return mcOut;
end

function sum2(MC1::mc, MC2::mc) ## Neither MC1 nor MC2 constant
    assert(MC1.nsub == MC2.nsub);
    mcOut = mc(MC1.nsub);
    mcOut.I = MC1.I + MC2.I;
    mcOut.cv = MC1.cv + MC2.cv;
    mcOut.cc = MC1.cc + MC2.cc;
    mcOut.cvsub = MC1.cvsub + MC2.cvsub;
    mcOut.ccsub = MC1.ccsub + MC2.ccsub;
    return mcOut;
end

import Base.+
function +(MC1::mc,MC2::mc)
    if (MC2.constant)
        return sum1(MC1, MC2);
    end
    if (MC1.constant)
        return sum1(MC2,MC1);
    end

    return sum2(MC1,MC2);
end

## Subtraction
import Base.-
function -(MC1::mc,x::Real)
    mcOut = mc(MC1.nsub);
    mcOut.I = MC1.I - x;
    mcOut.cv = MC1.cv - x;
    mcOut.cc = MC1.cc - x;
    mcOut.cvsub = MC1.cvsub;
    mcOut.ccsub = MC1.ccsub;
    return mcOut;
end

import Base.-
function -(x::Real,MC1::mc)
    return -(MC1-x);
end

function sub1(MC1::mc, MC2::mc) ## MC2 constant
    mcOut = mc(MC1.nsub);
    mcOut.I = MC1.I - MC2.I;
    mcOut.cv = MC1.cv - MC2.cc;
    mcOut.cc = MC1.cc - MC2.cv;
    mcOut.cvsub = MC1.cvsub;
    mcOut.ccsub = MC1.ccsub;
```

```
    return mcOut;
end

function sub2(MC1::mc, MC2::mc) ## MC1 constant
    mcOut = mc(MC2.nsub);
    mcOut.I = MC1.I - MC2.I;
    mcOut.cv = MC1.cv - MC2.cc;
    mcOut.cc = MC1.cc - MC2.cv;
    mcOut.cvsub = -MC2.ccsub;
    mcOut.ccsub = -MC2.cvsub;
    return mcOut;
end

function sub3(MC1::mc, MC2::mc) ## Neither MC1 nor MC2 constant
    assert(MC1.nsub == MC2.nsub);
    mcOut = mc(MC1.nsub);
    mcOut.I = MC1.I - MC2.I;
    mcOut.cv = MC1.cv - MC2.cc;
    mcOut.cc = MC1.cc - MC2.cv;
    mcOut.cvsub = MC1.cvsub - MC2.ccsub;
    mcOut.ccsub = MC1.ccsub - MC2.cvsub;
    return mcOut;
end

import Base.-
function -(MC1::mc,MC2::mc)
    if (MC2.constant)
        return sub1(MC1, MC2);
    end
    if (MC1.constant)
        return sub2(MC1,MC2);
    end

    return sub3(MC1,MC2);
end

## Multiplication
import Base.*
function *(MC1::mc,x::Real)
    mcOut = mc(MC1.nsub);
    mcOut.I = MC1.I * x;

    if (x >= 0.0)
        mcOut.cv = MC1.cv * x;
        mcOut.cc = MC1.cc * x;
        mcOut.cvsub = MC1.cvsub * x;
        mcOut.ccsub = MC1.ccsub * x;
    else
        mcOut.cv = MC1.cc * x;
        mcOut.cc = MC1.cv * x;
        mcOut.cvsub = MC1.ccsub * x;
        mcOut.ccsub = MC1.cvsub * x;
    end

    return mcOut;
end
```

```
import Base.*
function *(x::Real,MC1::mc)
    return MC1*x;
end

function mul1_mc1pos_mc2pos(MC1::mc, MC2::mc) ## MC2 constant, both positive
    mcOut = mc(MC1.nsub);
    mcOut.I = MC1.I*MC2.I;

    cv1 = MC2.I.hi*MC1.cv+MC1.I.hi*MC2.cv-MC1.I.hi*MC2.I.hi;
    cv2 = MC2.I.lo*MC1.cv+MC1.I.lo*MC2.cv-MC1.I.lo*MC2.I.lo;

    if (cv1 > cv2)
        mcOut.cv = cv1;
        mcOut.cvsub = MC2.I.hi*MC1.cvsub;
    else
        mcOut.cv = cv2;
        mcOut.cvsub = MC2.I.lo*MC1.cvsub;
    end

    cc1 = MC2.I.lo*MC1.cc+MC1.I.hi*MC2.cc-MC1.I.hi*MC2.I.lo;
    cc2 = MC2.I.hi*MC1.cc+MC1.I.lo*MC2.cc-MC1.I.lo*MC2.I.hi;

    if (cc1 < cc2)
        mcOut.cc = cc1;
        mcOut.ccsub = MC2.I.lo*MC1.ccsub;
    else
        mcOut.cc = cc2;
        mcOut.ccsub = MC2.I.hi*MC1.ccsub
    end

    return mcOut;
end

function mul2_mc1pos_mc2pos(MC1::mc, MC2::mc) ## Neither constant, both positi
ve
    assert(MC1.nsub==MC2.nsub)
    mcOut = mc(MC1.nsub);

        mcOut.I = MC1.I*MC2.I;

    cv1 = MC2.I.hi*MC1.cv+MC1.I.hi*MC2.cv-MC1.I.hi*MC2.I.hi;
    cv2 = MC2.I.lo*MC1.cv+MC1.I.lo*MC2.cv-MC1.I.lo*MC2.I.lo;

    if (cv1 > cv2)
        mcOut.cv = cv1;
        mcOut.cvsub = MC2.I.hi*MC1.cvsub + MC1.I.hi*MC2.cvsub;
    else
        mcOut.cv = cv2;
        mcOut.cvsub = MC2.I.lo*MC1.cvsub + MC1.I.lo*MC2.cvsub;
    end

    cc1 = MC2.I.lo*MC1.cc+MC1.I.hi*MC2.cc-MC1.I.hi*MC2.I.lo;
    cc2 = MC2.I.hi*MC1.cc+MC1.I.lo*MC2.cc-MC1.I.lo*MC2.I.hi;

    if (cc1 < cc2)
        mcOut.cc = cc1;
```

```
        mcOut.ccsub = MC2.I.lo*MC1.ccsub + MC1.I.hi*MC2.ccsub;
    else
        mcOut.cc = cc2;
        mcOut.ccsub = MC2.I.hi*MC1.ccsub + MC1.I.lo*MC2.ccsub;
    end

    return mcOut;
end

function mul1_mc1pos_mc2mix(MC1::mc, MC2::mc) ## MC2 constant, 0 in MC2
    mcOut = mc(MC1.nsub);
    mcOut.I = MC1.I*MC2.I;

    cv1 = MC2.I.hi*MC1.cv+MC1.I.hi*MC2.cv-MC1.I.hi*MC2.I.hi;
    cv2 = MC2.I.lo*MC1.cc+MC1.I.lo*MC2.cv-MC1.I.lo*MC2.I.lo;

    if (cv1 > cv2)
        mcOut.cv = cv1;
        mcOut.cvsub = MC2.I.hi*MC1.cvsub;
    else
        mcOut.cv = cv2;
        mcOut.cvsub = MC2.I.lo*MC1.ccsub;
    end

    cc1 = MC2.I.lo*MC1.cv+MC1.I.hi*MC2.cc-MC1.I.hi*MC2.I.lo;
    cc2 = MC2.I.hi*MC1.cc+MC1.I.lo*MC2.cc-MC1.I.lo*MC2.I.hi;

    if (cc1 < cc2)
        mcOut.cc = cc1;
        mcOut.ccsub = MC2.I.lo*MC1.cvsub;
    else
        mcOut.cc = cc2;
        mcOut.ccsub = MC2.I.hi*MC1.ccsub;
    end
    return mcOut;
end

function mul2_mc1pos_mc2mix(MC1::mc, MC2::mc) ## MC1 constant, 0 in MC2
    mcOut = mc(MC2.nsub);
    mcOut.I = MC1.I*MC2.I;

    cv1 = MC2.I.hi*MC1.cv+MC1.I.hi*MC2.cv-MC1.I.hi*MC2.I.hi;
    cv2 = MC2.I.lo*MC1.cc+MC1.I.lo*MC2.cv-MC1.I.lo*MC2.I.lo;

    if (cv1 > cv2)
        mcOut.cv = cv1;
        mcOut.cvsub = MC1.I.hi*MC2.cvsub;
    else
        mcOut.cv = cv2;
        mcOut.cvsub = MC1.I.lo*MC2.cvsub;
    end

    cc1 = MC2.I.lo*MC1.cv+MC1.I.hi*MC2.cc-MC1.I.hi*MC2.I.lo;
    cc2 = MC2.I.hi*MC1.cc+MC1.I.lo*MC2.cc-MC1.I.lo*MC2.I.hi;

    if (cc1 < cc2)
        mcOut.cc = cc1;
```

```
        mcOut.ccsub = MC2.I.hi*MC2.ccsub;
    else
        mcOut.cc = cc2;
        mcOut.ccsub = MC2.I.lo*MC2.ccsub
    end
    return mcOut;
end

function mul3_mc1pos_mc2mix(MC1::mc, MC2::mc) ## Neither constant, 0 in MC2
    assert(MC1.nsub==MC2.nsub)
    mcOut = mc(MC1.nsub);
    mcOut.I = MC1.I*MC2.I;

    cv1 = MC2.I.hi*MC1.cv+MC1.I.hi*MC2.cv-MC1.I.hi*MC2.I.hi;
    cv2 = MC2.I.lo*MC1.cc+MC1.I.lo*MC2.cv-MC1.I.lo*MC2.I.lo;

    if (cv1 > cv2)
        mcOut.cv = cv1;
        mcOut.cvsub = MC2.I.hi*MC1.cvsub+MC1.I.hi*MC2.cvsub;
    else
        mcOut.cv = cv2;
        mcOut.cvsub = MC2.I.lo*MC1.ccsub+MC1.I.lo*MC2.cvsub;
    end

    cc1 = MC2.I.lo*MC1.cv+MC1.I.hi*MC2.cc-MC1.I.hi*MC2.I.lo;
    cc2 = MC2.I.hi*MC1.cc+MC1.I.lo*MC2.cc-MC1.I.lo*MC2.I.hi;

    if (cc1 < cc2)
        mcOut.cc = cc1;
        mcOut.ccsub = MC2.I.lo*MC1.cvsub+MC1.I.hi*MC2.ccsub;
    else
        mcOut.cc = cc2;
        mcOut.ccsub = MC2.I.hi*MC1.ccsub+MC1.I.lo*MC2.ccsub;
    end
    return mcOut;
end

function mul1_mc1mix_mc2mix(MC1::mc, MC2::mc) ## MC2 constant, 0 in both
    mcOut = mc(MC1.nsub);
    mcOut.I = MC1.I*MC2.I;

    cv1 = MC2.I.hi*MC1.cv+MC1.I.hi*MC2.cv-MC1.I.hi*MC2.I.hi;
    cv2 = MC2.I.lo*MC1.cc+MC1.I.lo*MC2.cc-MC1.I.lo*MC2.I.lo;

    if (cv1 > cv2)
        mcOut.cv = cv1;
        mcOut.cvsub = MC2.I.hi*MC1.cvsub;
    else
        mcOut.cv = cv2;
        mcOut.cvsub = MC2.I.lo*MC1.ccsub;
    end

    cc1 = MC2.I.lo*MC1.cv+MC1.I.hi*MC2.cc-MC1.I.hi*MC2.I.lo;
    cc2 = MC2.I.hi*MC1.cc+MC1.I.lo*MC2.cv-MC1.I.lo*MC2.I.hi;

    if (cc1 < cc2)
        mcOut.cc = cc1;
```

```
        mcOut.ccsub = MC2.I.lo*MC1.cvsub;
    else
        mcOut.cc = cc2;
        mcOut.ccsub = MC2.I.hi*MC1.ccsub;
    end
    return mcOut;
end

function mul2_mc1mix_mc2mix(MC1::mc, MC2::mc) ## Neither constant, 0 in both
    assert(MC1.nsub==MC2.nsub)
    mcOut = mc(MC1.nsub);
    mcOut.I = MC1.I*MC2.I;

    cv1 = MC2.I.hi*MC1.cv+MC1.I.hi*MC2.cv-MC1.I.hi*MC2.I.hi;
    cv2 = MC2.I.lo*MC1.cc+MC1.I.lo*MC2.cc-MC1.I.lo*MC2.I.lo;

    if (cv1 > cv2)
        mcOut.cv = cv1;
        mcOut.cvsub = MC2.I.hi*MC1.cvsub+MC1.I.hi*MC2.cvsub;
    else
        mcOut.cv = cv2;
        mcOut.cvsub = MC2.I.lo*MC1.ccsub+MC1.I.lo*MC2.ccsub;
    end

    cc1 = MC2.I.lo*MC1.cv+MC1.I.hi*MC2.cc-MC1.I.hi*MC2.I.lo;
    cc2 = MC2.I.hi*MC1.cc+MC1.I.lo*MC2.cv-MC1.I.lo*MC2.I.hi;

    if (cc1 < cc2)
        mcOut.cc = cc1;
        mcOut.ccsub = MC2.I.lo*MC1.cvsub+MC1.I.hi*MC2.ccsub;
    else
        mcOut.cc = cc2;
        mcOut.ccsub = MC2.I.hi*MC1.ccsub+MC1.I.lo*MC2.cvsub;
    end
    return mcOut;
end

import Base.*
function *(MC1::mc,MC2::mc)
    if (MC1.I.lo >= 0.0)
        if (MC2.I.lo >= 0.0)
            if (MC2.constant)
                return cut(mul1_mc1pos_mc2pos(MC1,MC2));
            end
            if (MC1.constant)
                return cut(mul1_mc1pos_mc2pos(MC2,MC1));
            end
            return cut(mul2_mc1pos_mc2pos(MC1,MC2));
        end
        if (MC2.I.hi <= 0.0)
            return -(MC1*(-MC2));
        end
        if (MC2.constant)
            return cut(mul1_mc1pos_mc2mix(MC1,MC2));
        end
        if (MC1.constant)
            return cut(mul2_mc1pos_mc2mix(MC1,MC2));
```

```
        end
        return cut(mul3_mc1pos_mc2mix(MC1,MC2));
    end

    if (MC1.I.hi <= 0.0)
        if (MC2.I.lo >= 0.0)
            return -((-MC1)*MC2);
        end
        if (MC2.I.hi <= 0.0)
            return (-MC1) * (-MC2);
        end
        return -(MC2 * (-MC1));
    end

    if (MC2.I.lo >= 0.0)
        return MC2*MC1;
    end

    if (MC2.I.hi <= 0.0)
        return -((-MC2)*MC1);
    end

    if (MC2.constant)
        return cut(mul1_mc1mix_mc2mix(MC1,MC2));
    end
    if (MC1.constant)
        return cut(mul1_mc1mix_mc2mix(MC2,MC1));
    end

    return cut(mul2_mc1mix_mc2mix(MC1,MC2));
end

function inv(MC1::mc)
    assert(!(MC1.I.lo <= 0.0 && MC1.I.hi >= 0.0))
    mcOut = mc(MC1.nsub);
    mcOut.I = 1./MC1.I;
    if (MC1.I.lo > 0.)
        ## convex part
        imid = -1;
        vmid = mid(MC1.cv,MC1.cc,MC1.I.hi,imid);
        mcOut.cv = 1./vmid;
        for i=1:mcOut.nsub
            mcOut.cvsub[i] = -mid(MC1.cvsub,MC1.ccsub,i,imid)/(vmid^2);
        end
        ## concave part
        imid = -1;
        mcOut.cc = 1./MC1.I.lo + 1./MC1.I.hi - mid(MC1.cv,MC1.cc,MC1.I.lo,imid
(MC1.I.lo*MC1.I.hi);
        for i=1:mcOut.nsub
            mcOut.ccsub[i] = -mid(MC1.cvsub,MC1.ccsub,i,imid)/(MC1.I.lo*MC1.I.

        end
    else
        ## convex part
        imid = -1;
        mcOut.cv = 1./MC1.I.lo + 1./MC1.I.hi - mid(MC1.cv,MC1.cc,MC1.I.hi,imid
(MC1.I.lo*MC1.I.hi);
```

```
        for i=1:mcOut.nsub
            mcOut.cvsub[i] = -mid(MC1.cvsub,MC1.ccsub,i,imid)/(MC1.I.lo*MC1.I.
        end

        ## concave part
        imid = -1;
        vmid = mid(MC1.cv,MC1.cc,MC1.I.lo,imid);
        mcOut.cc = 1./vmid;
        for i=1:mcOut.nsub
            mcOut.ccsub[i] = -mid(MC1.cvsub,MC1.ccsub,i,imid)/(vmid^2);
        end
    end

    return cut(mcOut);
end

import Base./
function /(MC1::mc,x::Real)
    assert(x!=0.0)
    return (1.0 / x) * MC1;
end

function /(x::Real,MC1::mc)
    return x * inv(MC1);
end

function /(MC1::mc,MC2::mc)
    return MC1 * inv(MC2);
end

import Base.exp
function exp(MC1::mc)
    mcOut = mc(MC1.nsub);
    mcOut.I = exp(MC1.I);

    # Convex part
    imid = -1;
    mcOut.cv = exp(mid(MC1.cv, MC1.cc, MC1.I.lo, imid));
    for i=1:mcOut.nsub
        mcOut.cvsub[i] = mcOut.cv*mid(MC1.cvsub,MC1.ccsub,i,imid);
    end

    # Concave part
    imid = -1;
    r = 0.0;
    assert(MC1.I.lo!=MC1.I.hi)
    r = (exp(MC1.I.hi) - exp(MC1.I.lo))/(MC1.I.hi-MC1.I.lo);
    mcOut.cc = exp(MC1.I.hi) + r * ( mid(MC1.cv,MC1.cc,MC1.I.hi,imid)  - MC1.I
 );
    for i=1:mcOut.nsub
        mcOut.ccsub[i] = r*mid(MC1.cvsub,MC1.ccsub,i,imid);
    end

    return cut(mcOut);
end
```

```
import Base.log
function log(MC1::mc)
    mcOut = mc(MC1.nsub);
    mcOut.I = log(MC1.I);

    # Convex part
    imid = -1;
    r = 0.0;
    assert(MC1.I.lo!=MC1.I.hi)
    r = (log(MC1.I.hi) - log(MC1.I.lo))/(MC1.I.hi-MC1.I.lo);
    mcOut.cv = log(MC1.I.lo) + r * ( mid(MC1.cv,MC1.cc,MC1.I.lo,imid)  - MC1.I
);
    for i=1:mcOut.nsub
        mcOut.cvsub[i] = r*mid(MC1.cvsub,MC1.ccsub,i,imid);
    end

    # Concave part
    imid = -1;
    vmid = mid(MC1.cv, MC1.cc, MC1.I.hi, imid);
    mcOut.cc = log(vmid);
    for i=1:mcOut.nsub
        mcOut.ccsub[i] = mid(MC1.cvsub,MC1.ccsub,i,imid)/vmid;
    end

    return cut(mcOut);
end

import Base.sqrt
function sqrt(MC1::mc)
    mcOut = mc(MC1.nsub);
    mcOut.I = sqrt(MC1.I);
    assert(MC1.I.lo>=0.0)

    # Convex part
    imid = -1;
    r = 0.0;
    assert(MC1.I.lo!=MC1.I.hi)
    r = (sqrt(MC1.I.hi) - sqrt(MC1.I.lo))/(MC1.I.hi-MC1.I.lo);
    mcOut.cv = sqrt(MC1.I.lo) + r * ( mid(MC1.cv,MC1.cc,MC1.I.lo,imid)  - MC1.I
lo);
    for i=1:mcOut.nsub
        mcOut.cvsub[i] = r*mid(MC1.cvsub,MC1.ccsub,i,imid);
    end

    # Concave part
    imid = -1;
    vmid = mid(MC1.cv, MC1.cc, MC1.I.hi, imid);
    mcOut.cc = sqrt(vmid);
    for i=1:mcOut.nsub
        mcOut.ccsub[i] = mid(MC1.cvsub,MC1.ccsub,i,imid)/(2.0*mcOut.cc);
    end

    return cut(mcOut);

end

import Base.^
```

```
function ^(MC1::mc,n::Integer)

    if (n==-1)
        return inv(MC1);
    end

    if (n==0)
        return mc((1.0..1.0));
    end

    if (n==1)
        return MC1;
    end

    if (n==2)
        mcOut = mc(MC1.nsub);
        mcOut.I = MC1.I^n;

        mcOut = mc(MC1.nsub);
        mcOut.I = MC1.I^n;

        # Convex part
        imid = -1;
        zmin = mid(MC1.I.lo,MC1.I.hi,0.0);
        mcOut.cv = mid(MC1.cv,MC1.cc,zmin,imid)^n;
        for i=1:mcOut.nsub
            mcOut.cvsub[i] = n*mid(MC1.cvsub,MC1.ccsub,i,imid)*mid(MC1.cv,MC1.
zmin,imid)^(n-1);
        end

        # Concave part
        imid = -1;
        zmax = MC1.I.lo^n > MC1.I.hi^n ? MC1.I.lo : MC1.I.hi;
        r = (MC1.I.lo == MC1.I.hi)? 0.0 : (MC1.I.hi^n - MC1.I.lo^n)/(MC1.I.hi-M
I.lo);
        mcOut.cc = MC1.I.lo^n + r*(mid(MC1.cv,MC1.cc,zmax,imid) - MC1.I.lo);
        for i=1:mcOut.nsub
            mcOut.ccsub[i] = r*mid(MC1.cvsub,MC1.ccsub,i,imid);
        end
        return cut(mcOut);
    end

    if (n>=3)
        return MC1*(MC1^(n-1));
    end

    if (n<=-2)
        return inv(MC1)*(MC1^(n+1));
    end

end

import Base.^
function ^(MC1::mc,a::Real)
    return exp(a*log(MC1));
end
```

```
import Base.^
function ^(MC1::mc,MC2::mc)
    return exp(MC2*log(MC1));
end

import Base.^
function ^(a::Real,MC1::mc)
    return exp(MC1*log(a));
end
```

Out[10]: ^ (generic function with 77 methods)

Example:

Generating and plotting interval bounds and the McCormick relaxations for the function:

$$f(x) = \frac{x+4}{\exp(x)} + x^{2.5} + \sqrt{x} + \log(2x) - 0.1x^5$$

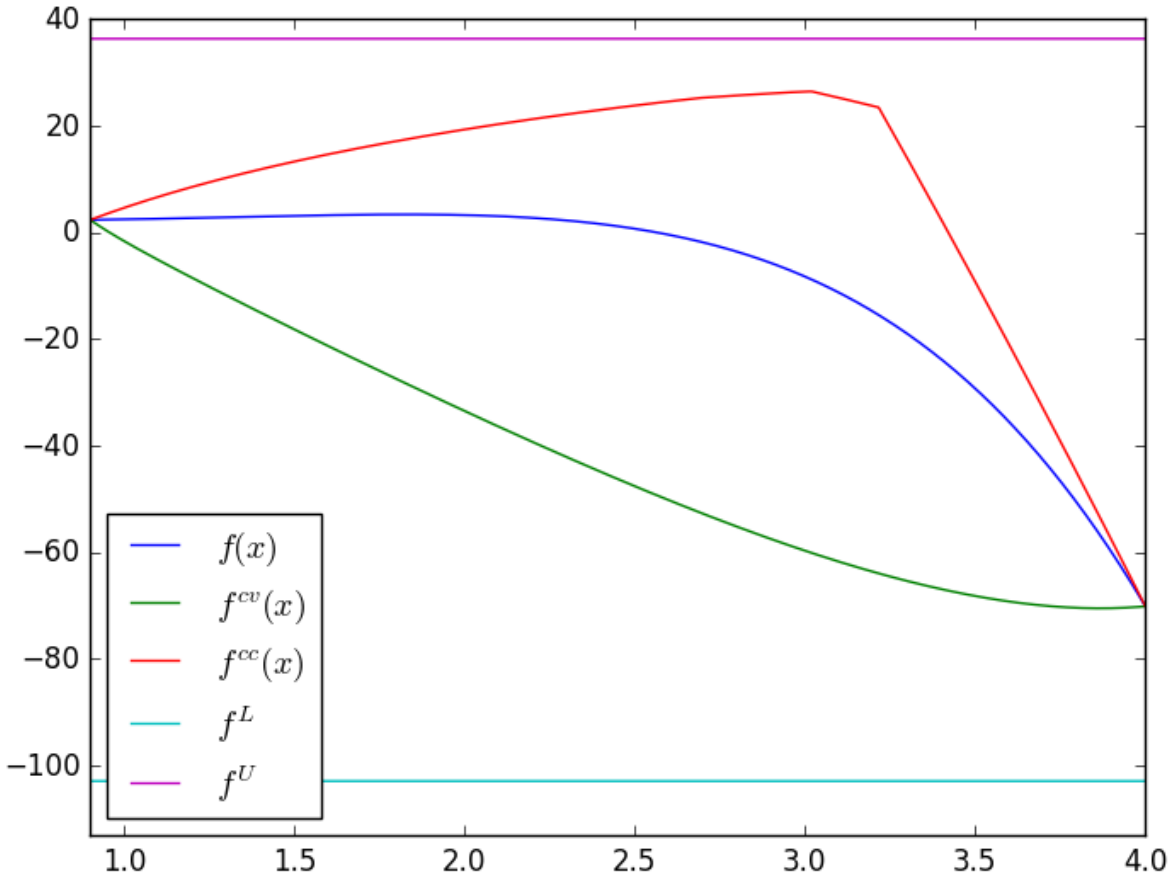
on the interval $X = [0.9, 4.0]$.

In [11]:
```
function mcPlot(f::Function,I::Interval)
    xx = linspace(I.lo,I.hi,2000);
    yI = f(I);
    y = Vector{Float64}(length(xx));
    ycv = Vector{Float64}(length(xx));
    ycc = Vector{Float64}(length(xx));
    for i=1:length(xx)
        y[i] = f(xx[i]);
        xmc = mc(I,xx[i]);
        ymc = f(xmc);
        ycv[i] = ymc.cv;
        ycc[i] = ymc.cc;
    end

    plot(xx,y)
    plot(xx,ycv)
    plot(xx,ycc)
    plot(xx,yI.lo*ones(length(xx)))
    plot(xx,yI.hi*ones(length(xx)))
    legend([L"f(x)",L"f^{cv}(x)",L"f^{cc}(x)",L"f^L",L"f^U"],loc=3)
    axis((I.lo,I.hi,yI.lo-0.1*abs(yI.lo),yI.hi+0.1*abs(yI.hi)));
end
```

Out[11]: mcPlot (generic function with 1 method)

In [12]:
```
f(x) = (x+4.0)/exp(x)+x^2.5-sqrt(x)+log(2*x)-0.1*x^5;
I = (0.9..4.0);
mcPlot(f,I);
```

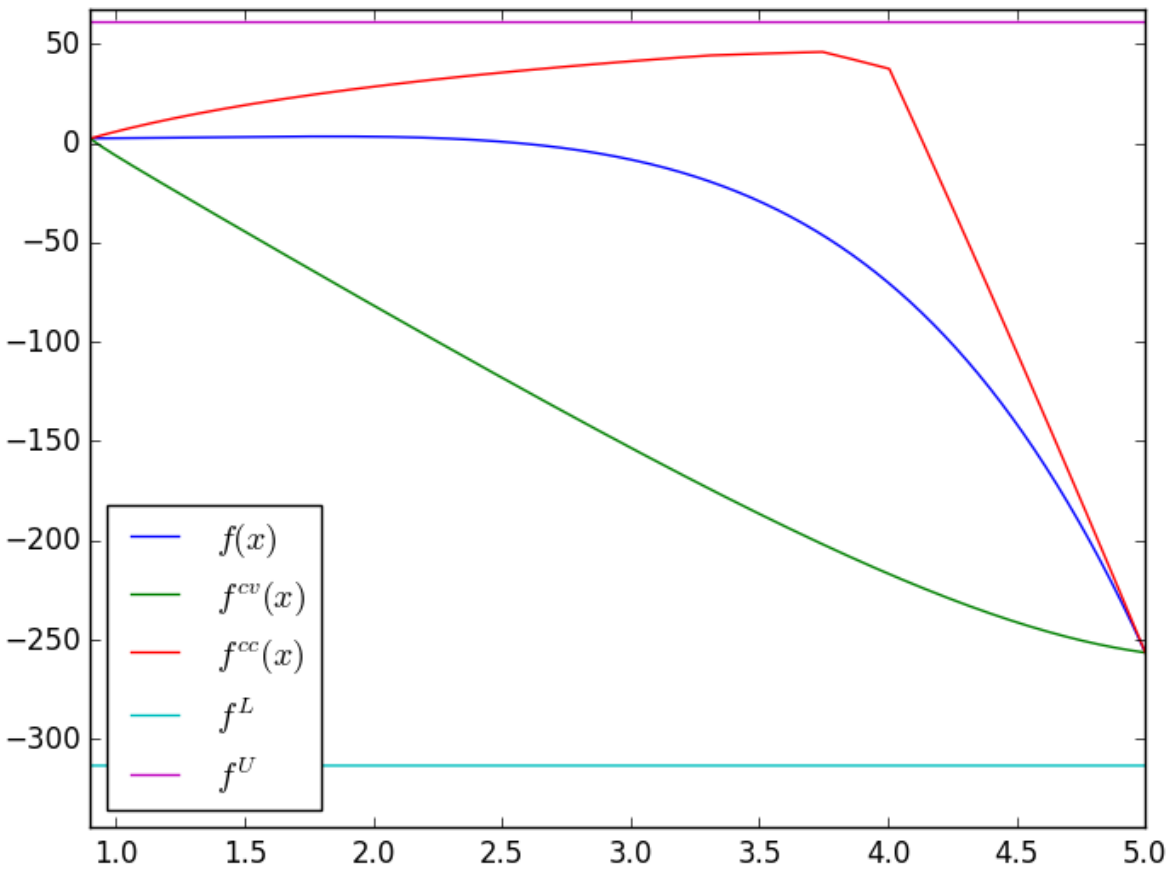


Note that the McCormick relaxations bound the function much more closely than the interval technique. However, the concave overestimator is not differentiable everywhere.

Note that if we change the box on which the intervals are constructed, they rapidly converge to the true range of the function:

In [13]:
```
fig=figure()
@manipulate for endpoint=1:10
    I = (0.9..endpoint)
    withfig(fig) do
        mcPlot(f,I)
    end
end
```

Out[13]:



Global optimization example, revisited

Now we resolve the previous problem using McCormick's technique to construct the relaxations. For the lower-bounding problem, the following convex program is constructed and solved at each node visited:

$$\min_{\mathbf{x}\in X} f^{cv}(\mathbf{x}) \text{ s.t. } \quad g^{cv}(\mathbf{x}) \leq \mathbf{0},\ h^{cv}(\mathbf{x}) \leq \mathbf{0} \leq h^{cc}(\mathbf{x}).$$

First we write new the lower-bounding problem and some wrappers for our objective function (and constraints if we have any) so that NLopt can solve for the lower bound.

In [14]:

```
## McCormick version of lower-bounding problem
function LB_problem2(probSize::Vector{Int64}, obj::Function,
conIneq::Function, conEq::Function, box::Vector{Interval{Float64}})

    nx = probSize[1];
    nIneq = probSize[2];
    nEq = probSize[3];

    opt = Opt(:LD_MMA, nx)

    lb = Vector{Float64}(nx);
    ub = Vector{Float64}(nx);
    x0 = Vector{Float64}(nx);

    for i=1:nx
        lb[i] = box[i].lo;
        ub[i] = box[i].hi;
        x0[i] = (lb[i] + ub[i])/2.0;
    end

    lower_bounds!(opt,lb);
    upper_bounds!(opt, ub);

    xtol_rel!(opt,1e-4);

    min_objective!(opt,(x,g)->objWrapperLB(x,g,lb,ub,obj));

    if nIneq > 0
        tol1 = 1.0e-6*ones(nIneq);
        inequality_constraint!(opt, (f,x,g)->conIneqWrapperLB(f,x,g,lb,ub,conI
neq),tol1);
    end
    if nEq > 0
        tol2 = 1.0e-6*ones(2*nEq);
        equality_constraint!(opt, (f,x,g)->conEqWrapperLB(f,x,g,lb,ub,conEq),t
ol2);
    end

    (solutionValue,solution,ret) = optimize!(opt, x0);

    status = (ret==:XTOL_REACHED ? true : false);

    return solution, solutionValue, status;
end

## Wrappers for providing objective and constraint info to NLopt
function objWrapperLB(x::Vector, grad::Vector, lb::Vector, ub::Vector, obj::Fu
nction)
    nx = length(x);
    xmc = Vector{mc}(nx);
    for i=1:nx
        xmc[i] = mc((lb[i]..ub[i]),x[i],nx);
    end
    for i=1:nx
        xmc[i].cvsub[i]=1.0;
        xmc[i].ccsub[i]=1.0;
    end
```

```
    mcFun = obj(xmc);
    if length(grad) > 0
        for i=1:nx
            grad[i] = mcFun.cvsub[i];
        end
    end
    return mcFun.cv;
end

function conIneqWrapperLB(result::Vector, x::Vector, grad::Matrix, lb::Vector,
 ub::Vector, conIneq::Function)
    nx = length(x);
    xmc = Vector{mc}(nx);
    for i=1:nx
        xmc[i] = mc((lb[i]..ub[i]),x[i],nx);
    end
    for i=1:nx
        xmc[i].cvsub[i]=1.0;
        xmc[i].ccsub[i]=1.0;
    end

    mcCon = conIneq(xmc);
    for i = 1:length(mcCon)
        result[i] = mcCon[i].cv;
    end
    if length(grad) > 0
        for i=1:length(mcCon)
            for j=1:nx
                grad[j,i] = mcCon[i].cvsub[j];
            end
        end
    end
end

function conEqWrapperLB(result::Vector, x::Vector, grad::Matrix, lb::Vector, u
b::Vector, conEq::Function)
    nx = length(x);
    xmc = Vector{mc}(nx);
    for i=1:nx
        xmc[i] = mc((lb[i]..ub[i]),x[i],nx);
    end
    for i=1:nx
        xmc[i].cvsub[i]=1.0;
        xmc[i].ccsub[i]=1.0;
    end

    mcCon = conEq(xmc);
    for i = 1:length(mcCon)
        result[i] = mcCon[i].cv;
    end
    for i = 1:length(mcCon)
        result[length(mcCon)+i] = -mcCon[i].cc;
    end

    if length(grad) > 0
        for i=1:length(mcCon)
```

```
            for j=1:nx
                grad[j,i] = mcCon[i].cvsub[j];
            end
        end
        for i=1:length(mcCon)
            for j=1:nx
                grad[j,length(mcCon)+i] = -mcCon[i].ccsub[j];
            end
        end
    end
end
```

```
WARNING: Base.ASCIIString is deprecated, use String instead.
  likely near C:\Users\Harry\.julia\v0.5\IJulia\src\kernel.jl:31
WARNING: Base.ASCIIString is deprecated, use String instead.
  likely near C:\Users\Harry\.julia\v0.5\IJulia\src\kernel.jl:31
WARNING: Base.ASCIIString is deprecated, use String instead.
```

```
Out[14]: conEqWrapperLB (generic function with 1 method)
```

Now, we're ready to optimize using the convex relaxations.

In [30]:
```
box = [(-3.0..3.0), (-3.0..3.0)];
opt = options("McCormick",1e-3,1e-8,10^5,10^5,0,0);
@time bab(sixHump, noConstraint, noConstraint, box, opt)
```

```
Iteration: 1
Nodes in memory: 2
Current solution found at: [0.0,0.0] with value = 0.0
Best possible solution value (lower bound): -1.0e50
The bounds are 0.0% converged.

Iteration: 84
Nodes in memory: 65
Current solution found at: [-0.126036,0.463529] with value = -0.6701878740052
957
Best possible solution value (lower bound): -6.215940192812888
The bounds are 0.0% converged.

Iteration: 88
Nodes in memory: 57
Current solution found at: [0.0,-0.874995] with value = -0.7177936208022256
Best possible solution value (lower bound): -5.586407306267247
The bounds are 0.0% converged.

Iteration: 91
Nodes in memory: 53
Current solution found at: [0.0,0.874995] with value = -0.7177936208028307
Best possible solution value (lower bound): -5.586407306267247
The bounds are 0.0% converged.

Iteration: 145
Nodes in memory: 69
Current solution found at: [-0.101247,0.578018] with value = -0.9076549049782
224
Best possible solution value (lower bound): -1.4203092643192115
The bounds are 48.7345640659011% converged.

Iteration: 157
Nodes in memory: 56
Current solution found at: [-0.0942407,0.578148] with value = -0.909240397545
308
Best possible solution value (lower bound): -1.1976329806447588
The bounds are 71.16074169005493% converged.

Iteration: 161
Nodes in memory: 44
Current solution found at: [-0.0938615,0.659625] with value = -1.009991122399
4242
Best possible solution value (lower bound): -1.1954031115446377
The bounds are 81.64221595287566% converged.

Iteration: 193
Nodes in memory: 51
Current solution found at: [-0.048236,0.703906] with value = -1.0245765543373
326
Best possible solution value (lower bound): -1.0906665850081398
The bounds are 93.54952732511506% converged.

Iteration: 197
Nodes in memory: 39
Current solution found at: [-0.0964883,0.703906] with value = -1.030778892909
6555
```

```
Best possible solution value (lower bound): -1.083651551239708
The bounds are 94.87061107927764% converged.

Iteration: 212
Nodes in memory: 40
Current solution found at: [-0.09375,0.703125] with value = -1.03079758901149
04
Best possible solution value (lower bound): -1.0508619207636927
The bounds are 98.05351390359348% converged.

Iteration: 225
Nodes in memory: 39
Current solution found at: [-0.0883791,0.703125] with value = -1.030899931366
8175
Best possible solution value (lower bound): -1.0526394761622724
The bounds are 97.89120707704078% converged.

Iteration: 249
Nodes in memory: 36
Current solution found at: [-0.0820777,0.714891] with value = -1.031334728340
642
Best possible solution value (lower bound): -1.03557826545696
The bounds are 99.5885392976977% converged.

Iteration: 264
Nodes in memory: 32
Current solution found at: [-0.0878916,0.714864] with value = -1.031569288593
0123
Best possible solution value (lower bound): -1.032872369278585
The bounds are 99.87367977120083% converged.

Best solution found at: [-0.0878916,0.714864] with value = -1.031569288593012
3
Best possible solution value (lower bound): -1.0322847544115328
The bounds are 99.93064297023652% converged.
Statistics: 272 iterations
  5.964958 seconds (37.70 M allocations: 1.393 GB, 7.50% gc time)
```

Due to the quadratic convergence rate of the relaxations, we can also tighten the termination tolerance dramatically and still solve the problem in a very similar amount of time (the cluster effect disappears).

In [16]:
```
opt = options("McCormick",1e-6,1e-8,10^5,10^5,0,0);
@time bab(sixHump, noConstraint, noConstraint, box, opt)
```

```
Iteration: 1
Nodes in memory: 2
Current solution found at: [0.0,0.0] with value = 0.0
Best possible solution value (lower bound): -1.0e50
The bounds are 0.0% converged.

Iteration: 84
Nodes in memory: 65
Current solution found at: [-0.126036,0.463529] with value = -0.6701878740052
957
Best possible solution value (lower bound): -6.215940192812888
The bounds are 0.0% converged.

Iteration: 88
Nodes in memory: 57
Current solution found at: [0.0,-0.874995] with value = -0.7177936208022256
Best possible solution value (lower bound): -5.586407306267247
The bounds are 0.0% converged.

Iteration: 91
Nodes in memory: 53
Current solution found at: [0.0,0.874995] with value = -0.7177936208028307
Best possible solution value (lower bound): -5.586407306267247
The bounds are 0.0% converged.

Iteration: 145
Nodes in memory: 69
Current solution found at: [-0.101247,0.578018] with value = -0.9076549049782
224
Best possible solution value (lower bound): -1.4203092643192115
The bounds are 48.7345640659011% converged.

Iteration: 157
Nodes in memory: 56
Current solution found at: [-0.0942407,0.578148] with value = -0.909240397545
308
Best possible solution value (lower bound): -1.1976329806447588
The bounds are 71.16074169005493% converged.

Iteration: 161
Nodes in memory: 44
Current solution found at: [-0.0938615,0.659625] with value = -1.009991122399
4242
Best possible solution value (lower bound): -1.1954031115446377
The bounds are 81.64221595287566% converged.

Iteration: 193
Nodes in memory: 51
Current solution found at: [-0.048236,0.703906] with value = -1.0245765543373
326
Best possible solution value (lower bound): -1.0906665850081398
The bounds are 93.54952732511506% converged.

Iteration: 197
Nodes in memory: 39
Current solution found at: [-0.0964883,0.703906] with value = -1.030778892909
6555
```

```
Best possible solution value (lower bound): -1.083651551239708
The bounds are 94.87061107927764% converged.

Iteration: 212
Nodes in memory: 40
Current solution found at: [-0.09375,0.703125] with value = -1.03079758901149
04
Best possible solution value (lower bound): -1.0508619207636927
The bounds are 98.05351390359348% converged.

Iteration: 225
Nodes in memory: 39
Current solution found at: [-0.0883791,0.703125] with value = -1.030899931366
8175
Best possible solution value (lower bound): -1.0526394761622724
The bounds are 97.89120707704078% converged.

Iteration: 249
Nodes in memory: 36
Current solution found at: [-0.0820777,0.714891] with value = -1.031334728340
642
Best possible solution value (lower bound): -1.03557826545696
The bounds are 99.5885392976977% converged.

Iteration: 264
Nodes in memory: 32
Current solution found at: [-0.0878916,0.714864] with value = -1.031569288593
0123
Best possible solution value (lower bound): -1.032872369278585
The bounds are 99.87367977120083% converged.

Iteration: 289
Nodes in memory: 34
Current solution found at: [-0.0908203,0.711915] with value = -1.031619495480
0861
Best possible solution value (lower bound): -1.0318732414841025
The bounds are 99.97540313990496% converged.

Iteration: 297
Nodes in memory: 27
Current solution found at: [-0.0893555,0.711915] with value = -1.031623389379
3771
Best possible solution value (lower bound): -1.0317152265860792
The bounds are 99.99109779715664% converged.

Iteration: 312
Nodes in memory: 27
Current solution found at: [-0.0900879,0.713379] with value = -1.031624116066
7508
Best possible solution value (lower bound): -1.0316468943572534
The bounds are 99.99779199709003% converged.

Iteration: 313
Nodes in memory: 18
Current solution found at: [-0.0900879,0.712646] with value = -1.031628214566
148
Best possible solution value (lower bound): -1.0316469043804501
```

```
The bounds are 99.99818831881115% converged.

Iteration: 325
Nodes in memory: 21
Current solution found at: [-0.0897217,0.712646] with value = -1.031628397424
3752
Best possible solution value (lower bound): -1.0316342095992153
The bounds are 99.99943660189517% converged.

Best solution found at: [-0.0897217,0.712646] with value = -1.031628397424375
2
Best possible solution value (lower bound): -1.0316288138747958
The bounds are 99.99995963174128% converged.
Statistics: 344 iterations
  6.348666 seconds (37.93 M allocations: 1.402 GB, 8.37% gc time)
```

Adding Constraints

Let's try adding the inequality constraint:

$$x_2 - x_1 \leq 0$$

to the previous optimization. This should give us the global minima with the same value as the one we've found so far but with positive x_1 and negative x_2.

Note that the local solver doesn't seem to like constrained problems very much and failed for a lot of the other tests I tried. I think I need to look at more options or potentially switch to using something more robust like IPOPT.

In [35]:
```
function x2leqx1(x::Vector);
    return [x[2]-x[1]];
end

box = [(-3.0..3.0), (-3.0..3.0)]
opt = options("McCormick",1e-6,1e-8,10^5,10^5,1,0);
@time bab(sixHump, x2leqx1, noConstraint, box, opt)
```

```
Iteration: 1
Nodes in memory: 2
Current solution found at: [0.0,0.0] with value = 0.0
Best possible solution value (lower bound): -1.0e50
The bounds are 0.0% converged.
```

```
WARNING: Method definition x2leqx1(Array{T<:Any, 1}) in module Main at In[3
3]:1 overwritten at In[35]:1.
```

```
Iteration: 54
Nodes in memory: 42
Current solution found at: [0.235576,-0.985571] with value = -0.1279175676480
2617
Best possible solution value (lower bound): -6.215940192812888
The bounds are 0.0% converged.

Iteration: 55
Nodes in memory: 34
Current solution found at: [0.126036,-0.463529] with value = -0.6701878740052
957
Best possible solution value (lower bound): -6.215940192812888
The bounds are 0.0% converged.

Iteration: 56
Nodes in memory: 30
Current solution found at: [0.0,-0.874995] with value = -0.7177936208022256
Best possible solution value (lower bound): -5.586407306267247
The bounds are 0.0% converged.

Iteration: 88
Nodes in memory: 39
Current solution found at: [0.101247,-0.578018] with value = -0.9076549049782
224
Best possible solution value (lower bound): -1.4203092643192115
The bounds are 48.7345640659011% converged.

Iteration: 94
Nodes in memory: 29
Current solution found at: [0.0942407,-0.578148] with value = -0.909240397545
308
Best possible solution value (lower bound): -1.1976329806447588
The bounds are 71.16074169005493% converged.

Iteration: 96
Nodes in memory: 23
Current solution found at: [0.0938615,-0.659625] with value = -1.009991122399
4242
Best possible solution value (lower bound): -1.1954031115446377
The bounds are 81.64221595287566% converged.

Iteration: 112
Nodes in memory: 26
Current solution found at: [0.048236,-0.703906] with value = -1.0245765543373
326
Best possible solution value (lower bound): -1.0906665850081398
The bounds are 93.54952732511506% converged.

Iteration: 114
Nodes in memory: 20
Current solution found at: [0.0964883,-0.703906] with value = -1.030778892909
6555
Best possible solution value (lower bound): -1.083651551239708
The bounds are 94.87061107927764% converged.

Iteration: 123
Nodes in memory: 21
```

```
Current solution found at: [0.09375,-0.703125] with value = -1.03079758901149
04
Best possible solution value (lower bound): -1.0508619207636927
The bounds are 98.05351390359348% converged.

Iteration: 131
Nodes in memory: 22
Current solution found at: [0.0883791,-0.703125] with value = -1.030899931366
8175
Best possible solution value (lower bound): -1.0526394761622724
The bounds are 97.89120707704078% converged.

Iteration: 143
Nodes in memory: 20
Current solution found at: [0.0820777,-0.714891] with value = -1.031334728340
642
Best possible solution value (lower bound): -1.03557826545696
The bounds are 99.5885392976977% converged.

Iteration: 149
Nodes in memory: 16
Current solution found at: [0.0878916,-0.714864] with value = -1.031569288593
0123
Best possible solution value (lower bound): -1.032872369278585
The bounds are 99.87367977120083% converged.

Iteration: 160
Nodes in memory: 17
Current solution found at: [0.0908203,-0.711915] with value = -1.031619495480
0861
Best possible solution value (lower bound): -1.0318732414841025
The bounds are 99.97540313990496% converged.

Iteration: 164
Nodes in memory: 13
Current solution found at: [0.0893555,-0.711915] with value = -1.031623389379
3771
Best possible solution value (lower bound): -1.0317152265860792
The bounds are 99.99109779715664% converged.

Iteration: 173
Nodes in memory: 14
Current solution found at: [0.0900879,-0.713379] with value = -1.031624116066
7508
Best possible solution value (lower bound): -1.0316468943572534
The bounds are 99.99779199709003% converged.

Iteration: 175
Nodes in memory: 11
Current solution found at: [0.0900879,-0.712646] with value = -1.031628214566
148
Best possible solution value (lower bound): -1.0316469043804501
The bounds are 99.99818831881115% converged.

Iteration: 178
Nodes in memory: 10
Current solution found at: [0.0897217,-0.712646] with value = -1.031628397424
```

```
3752
Best possible solution value (lower bound): -1.0316342095992153
The bounds are 99.99943660189517% converged.

Best solution found at: [0.0897217,-0.712646] with value = -1.031628397424375
2
Best possible solution value (lower bound): -1.0316288138747958
The bounds are 99.99995963174128% converged.
Statistics: 188 iterations
  4.128166 seconds (23.83 M allocations: 900.088 MB, 7.02% gc time)
```

In principle, we could also add equality constraints (and the code is written to accept them -- however, the solver in NLopt that works well with the nondifferentiable relaxations doesn't support them, and the other solvers do not seem very reliable.

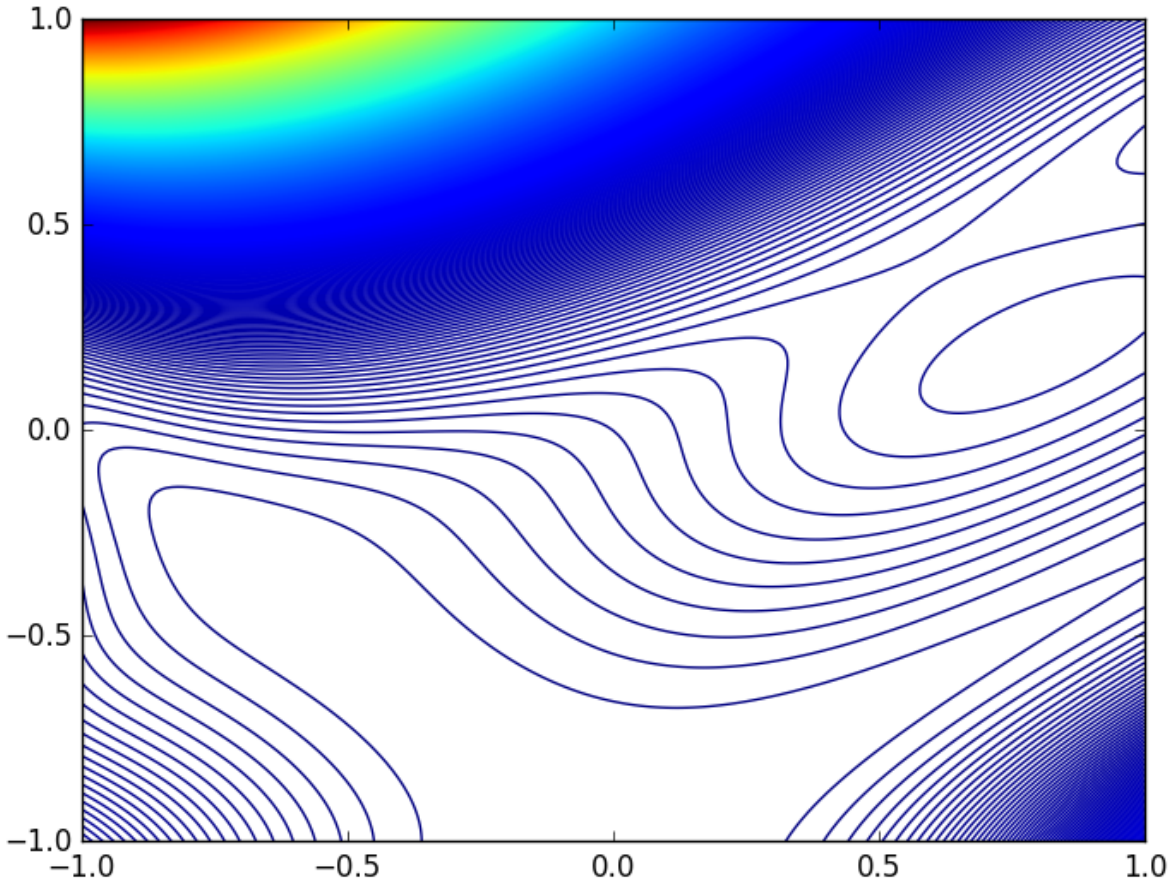
Other test problems:

Goldstein-Price function

In [19]:
```
function goldPrice(x::Vector)
    return (1+(x[1]+x[2]+1)^2*(19-14*x[1]+3*x[1]^2-14*x[2]+6*x[1]*x[2]+3*x[2]^
*
 (30+(2*x[1]-3*x[2])^2*(18-32*x[1]+12*x[1]^2+48*x[2]-36*x[1]*x[2]+27*x[2]^2));
end

box3 = [(-2.0..2.0), (-2.0..2.0)]
opt = options("McCormick",1e-6,1e-8,10^5,10^5,0,0);

## Plot the contours
xx = collect(-1.0:0.01:1.0);
zz = Matrix(length(xx),length(xx))
for i=1:length(xx)
    for j=1:length(xx)
        zz[i,j] = goldPrice([xx[i],xx[j]])
    end
end
contour(xx,xx,zz',1000)
```



Out[19]: PyObject <matplotlib.contour.QuadContourSet object at 0x000000002FB82128>

In [20]:
```
@time bab(goldPrice, noConstraint, noConstraint, box3, opt)
```

```
Iteration: 1
Nodes in memory: 2
Current solution found at: [0.0,0.0] with value = 600.0
Best possible solution value (lower bound): -1.0e50
The bounds are 0.0% converged.

Iteration: 2
Nodes in memory: 3
Current solution found at: [-1.0,0.0] with value = 278.0
Best possible solution value (lower bound): -8.788132e7
The bounds are 0.0% converged.

Iteration: 41
Nodes in memory: 41
Current solution found at: [0.547312,-0.451858] with value = 240.798799014584
38
Best possible solution value (lower bound): -1.4809097474601008e6
The bounds are 0.0% converged.

Iteration: 84
Nodes in memory: 80
Current solution found at: [1.75,0.25] with value = 138.578125
Best possible solution value (lower bound): -571826.0
The bounds are 0.0% converged.

Iteration: 139
Nodes in memory: 125
Current solution found at: [1.875,0.25] with value = 84.939697265625
Best possible solution value (lower bound): -175259.375
The bounds are 0.0% converged.

Iteration: 157
Nodes in memory: 134
Current solution found at: [0.763252,-0.468261] with value = 83.4389544175311
1
Best possible solution value (lower bound): -139463.7675429494
The bounds are 0.0% converged.

Iteration: 347
Nodes in memory: 298
Current solution found at: [-0.279276,-0.578757] with value = 57.769462592969
86
Best possible solution value (lower bound): -33001.516584819234
The bounds are 0.0% converged.

Iteration: 459
Nodes in memory: 382
Current solution found at: [-0.194587,-1.18267] with value = 28.9643063039160
47
Best possible solution value (lower bound): -18436.298410095133
The bounds are 0.0% converged.

Iteration: 920
Nodes in memory: 721
Current solution found at: [0.2397,-0.789565] with value = 24.805733103376227
Best possible solution value (lower bound): -4526.789980685484
The bounds are 0.0% converged.
```

```
Iteration: 1395
Nodes in memory: 1007
Current solution found at: [-0.207796,-1.07933] with value = 17.9650487921234
6
Best possible solution value (lower bound): -1890.3655820351005
The bounds are 0.0% converged.

Iteration: 1709
Nodes in memory: 1122
Current solution found at: [-0.0850719,-1.08691] with value = 7.1814009056223
36
Best possible solution value (lower bound): -1176.627886296616
The bounds are 0.0% converged.

Iteration: 2212
Nodes in memory: 1364
Current solution found at: [0.102136,-0.915758] with value = 6.54875058307156
7
Best possible solution value (lower bound): -596.5573836536
The bounds are 0.0% converged.

Iteration: 2874
Nodes in memory: 1632
Current solution found at: [-0.0617599,-0.940489] with value = 5.759375584386
9
Best possible solution value (lower bound): -277.8290862955896
The bounds are 0.0% converged.

Iteration: 3399
Nodes in memory: 1730
Current solution found at: [-0.0990915,-1.03788] with value = 5.6075530225155
17
Best possible solution value (lower bound): -159.30496312642384
The bounds are 0.0% converged.

Iteration: 3620
Nodes in memory: 1630
Current solution found at: [0.102348,-0.962506] with value = 5.48286971053659
1
Best possible solution value (lower bound): -123.29037931903753
The bounds are 0.0% converged.

Iteration: 3811
Nodes in memory: 1548
Current solution found at: [-0.0416125,-1.0396] with value = 3.80283581533796
47
Best possible solution value (lower bound): -101.68065719730174
The bounds are 0.0% converged.

Iteration: 4088
Nodes in memory: 1505
Current solution found at: [0.044594,-0.960979] with value = 3.75459711895463
45
Best possible solution value (lower bound): -74.53103659323975
The bounds are 0.0% converged.
```

```
Iteration: 4865
Nodes in memory: 1571
Current solution found at: [-0.0180564,-1.04137] with value = 3.7252552976761
906
Best possible solution value (lower bound): -24.03056511496304
The bounds are 0.0% converged.

Iteration: 4928
Nodes in memory: 1222
Current solution found at: [0.0242778,-0.963128] with value = 3.5415752599429
61
Best possible solution value (lower bound): -20.639016245595883
The bounds are 0.0% converged.

Iteration: 5289
Nodes in memory: 1096
Current solution found at: [-0.0205455,-1.01897] with value = 3.1819327655578
61
Best possible solution value (lower bound): -8.629629151253937
The bounds are 0.0% converged.

Iteration: 5324
Nodes in memory: 868
Current solution found at: [0.0210305,-0.980995] with value = 3.1777171470732
44
Best possible solution value (lower bound): -7.743499852595363
The bounds are 0.0% converged.

Iteration: 5813
Nodes in memory: 864
Current solution found at: [-0.0115717,-1.01818] with value = 3.1344958223716
39
Best possible solution value (lower bound): -0.567689379631684
The bounds are 0.0% converged.

Iteration: 5826
Nodes in memory: 617
Current solution found at: [0.00811595,-0.984322] with value = 3.094588143204
371
Best possible solution value (lower bound): -0.3566564248449815
The bounds are 0.0% converged.

Iteration: 5993
Nodes in memory: 533
Current solution found at: [-0.00784445,-1.00786] with value = 3.029116843047
4986
Best possible solution value (lower bound): 1.226669688851846
The bounds are 40.49595154004467% converged.

Iteration: 6018
Nodes in memory: 407
Current solution found at: [0.00786283,-0.992217] with value = 3.028311693493
9955
Best possible solution value (lower bound): 1.5009634880331006
The bounds are 49.564365889342255% converged.

Iteration: 6211
```

```
Nodes in memory: 400
Current solution found at: [-0.00390706,-1.00783] with value = 3.023941978973
603
Best possible solution value (lower bound): 2.481805118143164
The bounds are 82.07184977092541% converged.

Iteration: 6218
Nodes in memory: 295
Current solution found at: [0.003923,-0.992214] with value = 3.02334149488786
Best possible solution value (lower bound): 2.494556678197151
The bounds are 82.50992097370323% converged.

Iteration: 6307
Nodes in memory: 275
Current solution found at: [-0.00390466,-1.00391] with value = 3.007181180124
963
Best possible solution value (lower bound): 2.719804708792595
The bounds are 90.44365955627502% converged.

Iteration: 6308
Nodes in memory: 220
Current solution found at: [0.00391667,-0.996102] with value = 3.007102230208
935
Best possible solution value (lower bound): 2.727464556944665
The bounds are 90.70075934050168% converged.

Iteration: 6413
Nodes in memory: 216
Current solution found at: [-0.00195298,-1.00391] with value = 3.005940520085
785
Best possible solution value (lower bound): 2.9010116029342154
The bounds are 96.50928165576028% converged.

Iteration: 6414
Nodes in memory: 161
Current solution found at: [0.00195524,-0.996102] with value = 3.005863258103
8336
Best possible solution value (lower bound): 2.901972354559856
The bounds are 96.54372489287772% converged.

Iteration: 6459
Nodes in memory: 164
Current solution found at: [-0.00195253,-1.00195] with value = 3.001789984606
7326
Best possible solution value (lower bound): 2.943589528238824
The bounds are 98.06114162994872% converged.

Iteration: 6460
Nodes in memory: 143
Current solution found at: [0.001955,-0.99805] with value = 3.001779326992367
4
Best possible solution value (lower bound): 2.9442578378994164
The bounds are 98.08375357323203% converged.

Iteration: 6551
Nodes in memory: 149
Current solution found at: [-0.000976419,-1.00195] with value = 3.00148057704
```

```
97635
Best possible solution value (lower bound): 2.9789020644820856
The bounds are 99.24775416704942% converged.

Iteration: 6552
Nodes in memory: 107
Current solution found at: [0.000976938,-0.998049] with value = 3.00147059342
45445
Best possible solution value (lower bound): 2.978988410144874
The bounds are 99.2509610679204% converged.

Iteration: 6581
Nodes in memory: 103
Current solution found at: [-0.000976381,-1.00098] with value = 3.00044689076
1729
Best possible solution value (lower bound): 2.9874034256171123
The bounds are 99.56528258557826% converged.

Iteration: 6584
Nodes in memory: 90
Current solution found at: [0.000976907,-0.999024] with value = 3.00044548257
18834
Best possible solution value (lower bound): 2.987464457827999
The bounds are 99.56736341922276% converged.

Iteration: 6625
Nodes in memory: 102
Current solution found at: [-0.000488238,-1.00098] with value = 3.00036958559
0403
Best possible solution value (lower bound): 2.9949781911510875
The bounds are 99.82030898909228% converged.

Iteration: 6626
Nodes in memory: 90
Current solution found at: [0.000488329,-0.999024] with value = 3.00036836654
48313
Best possible solution value (lower bound): 2.9949865860189293
The bounds are 99.82062934051996% converged.

Iteration: 6655
Nodes in memory: 93
Current solution found at: [-0.000488235,-1.00049] with value = 3.00011164678
84974
Best possible solution value (lower bound): 2.996953884090332
The bounds are 99.89474516051608% converged.

Iteration: 6656
Nodes in memory: 82
Current solution found at: [0.000488382,-0.999512] with value = 3.00011147972
6387
Best possible solution value (lower bound): 2.996960931091059
The bounds are 99.89498561448073% converged.

Iteration: 6703
Nodes in memory: 99
Current solution found at: [-0.000244132,-1.00049] with value = 3.00009232713
7418
```

```
Best possible solution value (lower bound): 2.99877259540637
The bounds are 99.95601029611288% converged.

Iteration: 6704
Nodes in memory: 82
Current solution found at: [0.000244165,-0.999512] with value = 3.00009216751
6254
Best possible solution value (lower bound): 2.9987735766440125
The bounds are 99.95604832123097% converged.

Iteration: 6735
Nodes in memory: 86
Current solution found at: [-0.000244131,-1.00024] with value = 3.00002790325
7248
Best possible solution value (lower bound): 2.99925127505162
The bounds are 99.97411263392634% converged.

Iteration: 6736
Nodes in memory: 74
Current solution found at: [0.000244162,-0.999756] with value = 3.00002788168
0202
Best possible solution value (lower bound): 2.9992521290535805
The bounds are 99.97414181943579% converged.

Iteration: 6771
Nodes in memory: 91
Current solution found at: [-0.000122068,-1.00024] with value = 3.00002307400
31206
Best possible solution value (lower bound): 2.999696644720136
The bounds are 99.98911910758909% converged.

Iteration: 6772
Nodes in memory: 82
Current solution found at: [0.000122072,-0.999756] with value = 3.00002305545
1233
Best possible solution value (lower bound): 2.9996967604783187
The bounds are 99.98912358448975% converged.

Iteration: 6803
Nodes in memory: 86
Current solution found at: [-0.000122067,-1.00012] with value = 3.00000697485
0672
Best possible solution value (lower bound): 2.9998144121646355
The bounds are 99.99358125872203% converged.

Iteration: 6804
Nodes in memory: 74
Current solution found at: [0.000122074,-0.999878] with value = 3.00000697237
567
Best possible solution value (lower bound): 2.9998145116332484
The bounds are 99.99358465682934% converged.

Iteration: 6843
Nodes in memory: 89
Current solution found at: [-6.10345e-5,-1.00012] with value = 3.000005767692
5285
Best possible solution value (lower bound): 2.999924597183255
```

```
The bounds are 99.99729432155938% converged.

Iteration: 6844
Nodes in memory: 77
Current solution found at: [6.10361e-5,-0.999878] with value = 3.000005765510
6103
Best possible solution value (lower bound): 2.9999246131810207
The bounds are 99.99729492754572% converged.

Iteration: 6867
Nodes in memory: 80
Current solution found at: [-6.10347e-5,-1.00006] with value = 3.000001743561
686
Best possible solution value (lower bound): 2.9999538014932376
The bounds are 99.9984019319805% converged.

Iteration: 6868
Nodes in memory: 71
Current solution found at: [6.10357e-5,-0.999939] with value = 3.000001743293
2955
Best possible solution value (lower bound): 2.9999538147615077
The bounds are 99.9984023832021% converged.

Iteration: 6903
Nodes in memory: 85
Current solution found at: [-3.05175e-5,-1.00006] with value = 3.000001441787
392
Best possible solution value (lower bound): 2.9999812036968354
The bounds are 99.99932539730565% converged.

Iteration: 6904
Nodes in memory: 75
Current solution found at: [3.05177e-5,-0.999939] with value = 3.000001441568
6374
Best possible solution value (lower bound): 2.999981205932073
The bounds are 99.99932547910532% converged.

Iteration: 6923
Nodes in memory: 80
Current solution found at: [-3.05175e-5,-1.00003] with value = 3.000000435873
9325
Best possible solution value (lower bound): 2.999988475254307
The bounds are 99.99960131273707% converged.

Iteration: 6924
Nodes in memory: 75
Current solution found at: [3.05176e-5,-0.999969] with value = 3.000000435842
9284
Best possible solution value (lower bound): 2.9999884770206044
The bounds are 99.99960137264712% converged.

Iteration: 6971
Nodes in memory: 92
Current solution found at: [-1.52588e-5,-1.00003] with value = 3.000000360433
2273
Best possible solution value (lower bound): 2.9999953077295043
The bounds are 99.9998315765628% converged.
```